On Galois coverings and tilting modules

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Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field k. Let T be a basic tilting A-module with arbitrary finite projective dimension. For a fixed group G we compare the set of isoclasses of connected Galois coverings of A with group G and the set of isoclasses of connected Galois coverings of $End_A(T)$ with group G. Using the Hasse diagram $\overrightarrow{\mathcal{K}}_A$ (see [17] and [23]) of basic tilting A-modules, we give sufficient conditions on T under which there is a bijection between these two sets (these conditions are always verified when A is of finite representation type). Then we apply these results to study when the simple connectedness of A implies the one of $End_A(T)$ (see [5]). Finally, using an argument due to W. Crawley-Boevey, we prove that the type of any simply connected tilted algebra is a tree and that its first Hochschild cohomology group vanishes

Introduction

Let k be an algebraically closed field and let A be a finite dimensional k-algebra. In order to study the category $mod(A)$ of finite dimensional (left) A-modules we may assume that A is basic and connected. In the study of $mod(A)$, tilting theory has proved to be a powerful tool. Indeed, if T is a basic tilting A-module and if we set $B = End_A(T)$, then A and B have many common properties: Brenner-Butler Theorem establishes an equivalence between certain subcategories of $mod(A)$ and $mod(B)$ (see [10], [16] and [21]), A and B have equivalent derived categories (see [14]) and (in particular) they have isomorphic Grothendieck groups and isomorphic Hochschild cohomologies. In this text we will study the following problem relating A and B:

<div align="center">

is it possible to compare the Galois coverings of A and those of B? (P_1)

</div>

As an example, if $A = kQ$ with Q a finite quiver without oriented cycle and if T is an APR-tilting module associated to a sink x of Q (see [6]) then $B = kQ'$ where Q' is obtained from Q by reversing all the arrows endings at x. In particular Q and Q' have the same underlying graph and therefore A has a connected Galois covering with group G if and only if the same holds for B.

Recall that in order to consider Galois coverings of A we always consider A as a k-category. When $\mathcal{C} \to A$ is a Galois covering, it is possible to describe part of $mod(A)$ in terms of $\mathcal{C}$-modules (see for example [9] and [12]). This description is useful because $mod(\mathcal{C})$ is easier to study than $mod(A)$, especially when $\mathcal{C}$ is simply connected (this last situation may occur when A is of finite representation type, see [12]). Notice that simple connectedness and tilting theory have already been studied together through the following conjecture formulated in [5]:

<div align="center">

A is simply connected $\Longrightarrow$ B is simply connected (P_2)

</div>

More precisely, the above implication is true if: A is of finite representation type and T is of projective dimension at most one (see [2]), or if: $A = kQ$ (with Q a quiver) and B is tame (see [5], see also [3] for a generalisation to the case of quasi-tilted algebras). The two problems (P_1) and (P_2) are related because A is simply connected if and only if there is no proper Galois covering $\mathcal{C} \to A$ with $\mathcal{C}$ connected and locally bounded (see [19]).

In order to study the question (P_1) we will exhibit sufficient conditions for T to be of the first kind w.r.t. a fixed Galois covering $\mathcal{C} \xrightarrow{F} A$. Indeed, if T is of the first kind w.r.t. F, then it is possible to construct a Galois covering of B. Under additional hypotheses on T, the equivalence class of this Galois

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covering is uniquely determined by the equivalence class of F. Here we say that two Galois coverings $F \colon \mathcal{C} \to A$ and $F' \colon \mathcal{C}' \to A$ are equivalent if and only if there exists a commutative square of k-categories and k-linear functors:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sim} & \mathcal{C}' \\
\scriptstyle F \downarrow & & \downarrow \scriptstyle F' \\
A & \xrightarrow{\sim} & A
\end{array}
$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on the set of objects of A. For simplicity, let us say that A and B have the same connected Galois coverings with group G if there exists a bijection between the sets $Gal_A(G)$ and $Gal_B(G)$ where $Gal_A(G)$ (resp. $Gal_B(G)$) stands for the set of equivalence classes of Galois coverings $\mathcal{C} \to A$ (resp. $\mathcal{C} \to B$) with group G and with $\mathcal{C}$ connected and locally bounded. With this definition, we prove the following theorem which is the main result of this text and which partially answers (P_1):

Theorem 1. *Let T be a basic tilting A-module, let $B = End_A(T)$ and let G be group.*

1. *If $T' \in \overrightarrow{\mathcal{K}}_A$ lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing T, then $End_A(T)$ and $End_A(T')$ have the same connected Galois coverings with group G.*

2. *If T lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A or DA, then A and B have the same connected Galois coverings with group G.*

In particular, if $\overrightarrow{\mathcal{K}}_A$ is connected (which happens when A is of finite representation type) then A and B have the same connected Galois coverings with group G, for any group G.

Here $\overrightarrow{\mathcal{K}}_A$ is the Hasse diagram associated with the poset $\mathcal{T}_A$ of basic tilting A-modules (see [17] and [23]). Recall (see [12]) that when A is of finite representation type, A admits a connected Galois covering with group G if and only if G is a factor group of the fundamental group $\pi_1(A)$ of the Auslander-Reiten quiver of A with its mesh relations. Theorem 1 allows us to get the following corollary when A and B are of finite representation type. We thank Ibrahim Assem for having pointed out this corollary.

Corollary 1. *Let T be a basic tilting A-module and let $B = End_A(T)$. If both A and B are of finite representation type, then A and B have isomorphic fundamental groups.*

Theorem 1 also allows us to prove the following corollary related to (P_2).

Corollary 2. *(see [2] and [3]) Let T be a basic tilting A-module and let $B = End_A(T)$.*

1. *If $T' \in \overrightarrow{\mathcal{K}}_A$ lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing T, then: $End_A(T)$ is simply connected if and only if $End_A(T')$ is simply connected.*

2. *If T lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A or DA then: A is simply connected if and only if B is simply connected.*

In particular, f $\overrightarrow{\mathcal{K}}_A$ is connected (e.g. A is of finite representation type, see [17]), then: A is simply connected if and only if B is simply connected.

Theorem 1 uses the Hasse diagram $\overrightarrow{\mathcal{K}}_A$ in order to prove (in particular) that a basic tilting A-module lying in a specific connected component of $\overrightarrow{\mathcal{K}}_A$ is of the first kind w.r.t. a given connected Galois covering of A. On the other hand, an argument due to W. Crawley-Boevey (see [13]) proves that any rigid A-module M (i.e. such that $Ext_A^1(M, M) = 0$) is of the first kind w.r.t. any Galois covering of A with group $\mathbb{Z}$ if $car(k) = 0$ (resp. $\mathbb{Z}/p\mathbb{Z}$ if $car(k) = p$, p prime). Using this argument we are able to prove the following proposition. It gives a partial answer to (P_2) and to A. Skwroński's question in [24, Problem 1] wether it is true that a triangular algebra is simply connected if and only if its first Hochschild cohomology group vanishes.

Proposition 1. *(see [3], [5]) Assume that $A = kQ$ with Q a quiver without oriented cycle. Let T be a basic tilting A-module and let $B = End_A(T)$ (hence, B is tilted of type Q). Then the following implication holds:*

$$B \text{ is simply connected} \Longrightarrow Q \text{ is a tree and } HH^1(B) = 0$$

Recall that the implication of Proposition 1 has been proved in [5] for B tame tilted and in [3] for B tame quasi-tilted.

The text is organised as follows. In Section 1 we will give the definition of all the notions mentioned above and which will be used for the proof of Theorem 1. In Section 2 we will detail the construction and give some properties of the Galois covering F' of B starting from a Galois covering $F \colon \mathcal{C} \to A$ of A and

a basic tilting A-module T. In this study, we will introduce the following hypotheses on the A-module T: 1) T is of the first kind w.r.t. F (this ensures that F' exists), 2) the $\mathcal{C}$-module $F.T$ obtained from T by restricting the scalars is basic (this ensures that F' is connected if F is connected) 3) $\psi.N \simeq N$ for any direct summand N of T and for any automorphism $\psi \colon A \xrightarrow{\sim} A$ which restricts to the identity map on objects (this ensures that the equivalence class of F' does depend only on the equivalence class of F). These three hypotheses lack of simplicity, therefore, Section 3 is devoted to find simple sufficient conditions for the basic tilting A-module T to verify these. In particular, we will prove that the condition "T *lies in the connected component of* $\overrightarrow{\mathcal{K}}_A$ *containing* A" fits our requirements. Since our main objective is to establish a correspondence between the equivalence classes of the connected Galois coverings of A and those of B, we will need to find conditions for T to lie in both connected components of $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$ containing A and B respectively (recall that T is also a basic tilting B-module). This will be done in Section 4 where we compare the Hasse diagrams $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$. In particular, we will prove that there is an oriented path in $\overrightarrow{\mathcal{K}}_A$ starting at A and ending at T if and only if there is an oriented path in $\overrightarrow{\mathcal{K}}_B$ starting at B and ending at T. This equivalence will be used in Section 5 in order to prove Theorem 1, Corollary 1 and Corollary 2. Finally, in Section 6, we prove Proposition 1.

I would like to acknowledge Eduardo N. Marcos for his stimulating remarks concerning the implication (P_2) during the CIMPA school *Homological methods and representations of non-commutative algebras* in Mar del Plata, Argentina (February 2006).

1 Basic definitions and preparatory lemmata

Reminder on k-categories (see [9] for more details). A k-**category** is small category $\mathcal{C}$ such that for any $x, y \in Ob(\mathcal{C})$ the set ${}_y\mathcal{C}_x$ of morphisms from x to y is a k-vector space and such that the composition of morphisms in $\mathcal{C}$ is k-bilinear. A k-category $\mathcal{C}$ is called connected if and only if there is no non trivial partition $Ob(\mathcal{C}) = E \sqcup F$ such that ${}_y\mathcal{C}_x = {}_x\mathcal{C}_y = 0$ for any $x \in E, y \in F$.

All functors between k-categories are supposed to be k-linear. If $F \colon \mathcal{E} \to \mathcal{B}$ and $F' \colon \mathcal{E}' \to \mathcal{B}$ are functors between k-categories, then F and F' are called equivalent if there exists a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ \sim\ } & \mathcal{E}' \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\
\mathcal{B} & \xrightarrow{\ \sim\ } & \mathcal{B}
\end{array}
$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on $Ob(\mathcal{B})$. A **locally bounded** k-category is a k-category $\mathcal{C}$ verifying the following conditions:

  . distinct objects in $\mathcal{C}$ are not isomorphic,

  . for any $x \in Ob(\mathcal{C})$, the k-vector spaces $\bigoplus_{y \in Ob(\mathcal{C})} {}_y\mathcal{C}_x$ and $\bigoplus_{y \in Ob(\mathcal{C})} {}_x\mathcal{C}_y$ are finite dimensional,

  . for any $x \in Ob(\mathcal{C})$, the k-algebra ${}_x\mathcal{C}_x$ is local.

For example, let A be a basic finite dimensional k-algebra (basic means that A is the direct sum of pairwise non-isomorphic indecomposable projective A-modules) and let $\{e_1, \ldots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents. Then A can be viewed as a locally bounded k-category as follows: $e_1, \ldots, e_n$ are the objects of A, the space of morphisms from e_i to e_j is equal to $e_j A e_i$ for any i, j and the composition of morphisms is induced by the product in A. Notice that different choices for the primitive idempotents $e_1, \ldots, e_n$ give rise to isomorphic k-categories. In this text we shall always consider such an algebra A as a locally bounded k-category.

Modules over k-categories. If $\mathcal{C}$ is a k-category, a (left) $\mathcal{C}$-module is a k-linear functor $M \colon \mathcal{C} \to MOD(k)$ where $MOD(k)$ is the category of k-vector spaces. A morphism of $\mathcal{C}$-modules $M \to N$ is a k-linear natural transformation of functors. The category of $\mathcal{C}$-modules is denoted by $MOD(\mathcal{C})$.

A $\mathcal{C}$-module M is called locally finite dimensional (resp. finite dimensional) if and only if $M(x)$ is finite dimensional for any $x \in Ob(\mathcal{C})$ (resp. $\bigoplus_{x \in Ob(\mathcal{C})} M(x)$ is finite dimensional). The category of locally finite dimensional (resp. finite dimensional) $\mathcal{C}$-modules is denoted by $Mod(\mathcal{C})$ (resp. $mod(\mathcal{C})$). Notice that if $\mathcal{C} = A$ as above, then $Mod(\mathcal{C}) = mod(\mathcal{C})$.

We shall write $IND(\mathcal{C})$ (resp. $Ind(\mathcal{C})$, resp. $ind(\mathcal{C})$) for the full subcategory of $MOD(\mathcal{C})$ (resp. of $Mod(\mathcal{C})$, resp. of $mod(\mathcal{C})$) of indecomposable $\mathcal{C}$-modules. Finally, if $M = N_1 \bigoplus \ldots \bigoplus N_t$ with $N_i \in ind(\mathcal{C})$ for any i, then M is called **basic** if and only if $N_1, \ldots, N_t$ are pairwise non isomorphic.

Tilting modules. Let A be a basic finite dimensional k-algebra. A tilting A-module (see [10], [16] and [21]) is a module $T \in mod(A)$ verifying the following conditions:

($T1$) T has finite projective dimension (i.e. $pd_A(T) < \infty$),

($T2$) $Ext_A^i(T,T) = 0$ for any $i > 0$ (i.e. T is selforthogonal),

($T3$) there is an exact sequence in $mod(A)$: $0 \to A \to T_1 \to \ldots \to T_r \to 0$ with $T_1, \ldots, T_r \in add(T)$ (this last property means that $T_1, \ldots, T_r$ are direct sums of direct summands of T).

A module which satisfies conditions ($T1$) and ($T2$) above is called an exceptional module. Assume that T is a tilting A-module. Then, T is also a tilting $End_A(T)$-module for the following action: $f.t = f(t)$ for $f \in End_A(T)$ and $t \in T$. Assume moreover that T is basic as an A-module and fix a decomposition $T = T_1 \oplus \ldots \oplus T_n$ with $T_1, \ldots, T_n \in ind(A)$. This defines a decomposition of the unit of $End_A(T)$ into a sum of primitive pairwise orthogonal idempotents so that $B := End_A(T)$ is a locally bounded k-category as follows: the set of objects is $\{T_1, \ldots, T_n\}$ and for any i, j the space of morphisms $_{T_j}B_{T_i}$ is equal to $Hom_A(T_i, T_j)$. For any $x \in Ob(A)$, $T(x)$ is an indecomposable B-module:

$$
\begin{array}{ccc}
B & \to & MOD(k) \\
T_i \in Ob(B) & \mapsto & T_i(x) \\
u \in {}_{T_j}B_{T_i} & \mapsto & T_i(x) \xrightarrow{u_x} T_j(x)
\end{array}
$$

and $T = \bigoplus_{x \in Ob(A)} T(x)$. Finally, the following functor is an isomorphism of k-categories:

$$
\rho_A \colon \quad
\begin{array}{ccc}
A & \longrightarrow & End_B(T) \\
x \in Ob(A) & \longmapsto & T(x) \in Ob(End_B(T)) \\
u \in {}_yA_x & \longmapsto & T(x) \xrightarrow{T(u)} T(y)
\end{array}
$$

For more details on the above properties and for a more general study of $End_A(T)$, we refer the reader to [10], [14], [16] and [21].

Let $\mathcal{T}_A$ be the set of basic tilting A-modules up to isomorphism. Then $\mathcal{T}_A$ is endowed with a partial order introduced in [23] and defined as follows. If $T \in \mathcal{T}_A$, the right perpendicular category $T^\perp$ of T is defined by (see [7]):

$$
T^\perp = \{X \in mod(A) \mid (\forall i \geqslant 1) \ \ Ext_A^i(T,X) = 0\}
$$

If $T' \in \mathcal{T}_A$ is another basic tilting module, we write $T \leqslant T'$ provided that $T^\perp \subseteq T'^\perp$. In particular, we have $T \leqslant A$ for any $T \in \mathcal{T}_A$. In [17], D. Happel and L. Unger have proved that the Hasse diagram $\overrightarrow{\mathcal{K}}_A$ of $\mathcal{T}_A$ is as follows. The vertices in $\overrightarrow{\mathcal{K}}_A$ are the elements in $\mathcal{T}_A$ and there is an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_A$ if and only if: $T = X \bigoplus \overline{T}$ with $X \in ind(A)$, $T' = Y \bigoplus \overline{T}$ with $Y \in ind(A)$ and there exists a non split exact sequence $0 \to X \xrightarrow{u} M \xrightarrow{v} Y \to 0$ in $mod(A)$ with $M \in add(\overline{T})$. In such a situation, u (resp. v) is the left (resp. right) $add(\overline{T})$-approximation of X (resp. Y). For more details on $\overrightarrow{\mathcal{K}}_A$, we refer the reader to [17] and [18].

Galois coverings of k-categories. Let G be a group. A **free G-category** is a k-category $\mathcal{E}$ endowed with a morphism of groups $G \to Aut(\mathcal{E})$ such that the induced action of G on $Ob(\mathcal{E})$ is free. In this case, there exists a (unique) quotient $\mathcal{E} \to \mathcal{E}/G$ of $\mathcal{E}$ by G in the category of k-categories. With this property, a Galois covering of $\mathcal{B}$ with group G is by definition a functor $F \colon \mathcal{E} \to \mathcal{B}$ endowed with a group morphism $G \to Aut(F) = \{g \in Aut(\mathcal{E}) \mid F \circ g = F\}$ and verifying the following facts:

. the group morphism $G \to Aut(F) \hookrightarrow Aut(\mathcal{E})$ endows $\mathcal{E}$ with a structure of free G-category,

. the functor $\mathcal{E}/G \xrightarrow{\overline{F}} \mathcal{B}$ induced by F is an isomorphism.

This definition implies that the group morphism $G \to Aut(F)$ is one-to-one (actually one can show that this is an isomorphism when $\mathcal{E}$ is connected). Moreover for any $x \in Ob(\mathcal{B})$ the set $F^{-1}(x)$ is non empty and called the fiber of F at x. It verifies $F^{-1}(F(x)) = G.x$ for any $x \in Ob(\mathcal{E})$.

We recall that Galois coverings are particular cases of **covering functors** (see [9]). A covering functor is a k-linear functor $F \colon \mathcal{E} \to \mathcal{B}$ such that for any $x, y \in \mathcal{E}_0$, the following mappings induced by F are bijective:

$$
\bigoplus_{y' \in F^{-1}(F(y))} {}_{y'}\mathcal{E}_x \to {}_{F(y)}\mathcal{B}_{F(x)} \text{ and } \bigoplus_{x' \in F^{-1}(F(x))} {}_y\mathcal{E}_{x'} \to {}_{F(y)}\mathcal{B}_{F(x)}
$$

Remark that a covering functor is not supposed to restrict to a surjective mapping on objects. However, a covering functor is an isomorphism of k-categories if and only if it restricts to a bijective mapping on objects. Using basic linear algebra arguments it is easy to prove the following useful lemma:

Lemma 1.1. *Let p, q be k-linear functors such that the composition $q \circ p$ is defined. Then p, q and $q \circ p$ are covering functors as soon as two of them are so.*

If $F \colon \mathcal{E} \to \mathcal{B}$ is a Galois covering with group G and with $\mathcal{B}$ connected then $\mathcal{E}$ need not be connected. In such a case, if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$ where the $\mathcal{E}_i$'s are the connected components of $\mathcal{E}$, then for each i, the following functor:

$$F_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E} \to \mathcal{B}$$

is a Galois covering with group:

$$G_i := \{g \in G \mid g(Ob(\mathcal{E}_i)) \cap Ob(\mathcal{E}_i) \neq \emptyset\} = \{g \in G \mid g(Ob(\mathcal{E}_i)) = Ob(\mathcal{E}_i)\}$$

Moreover, if $i, j \in I$ then the groups G_i and G_j are conjugated in G and there exists a commutative diagram:

where the horizontal arrow is an isomorphism. This implies that G acts transitively on the set $\{\mathcal{E}_i \mid i \in I\}$ of the connected components of $\mathcal{E}$. Notice that all these facts may be false if $\mathcal{B}$ is not connected.

Two Galois coverings of $\mathcal{B}$ are called equivalent if and only if they are isomorphic as functors between k-categories (see above, this implies that the groups of the Galois coverings are isomorphic). The equivalence class of a Galois covering F will be denoted by $[F]$. Finally, we shall say for short that **a Galois covering $\mathcal{E} \to \mathcal{B}$ is connected** if and only if $\mathcal{E}$ is connected and locally bounded (this implies that $\mathcal{B}$ is connected and locally bounded, see [12, 1.2]).

Simply connected locally bounded k-categories. Let $\mathcal{B}$ be a locally bounded k-category. Then $\mathcal{B}$ is called simply connected if and only if there is no proper connected Galois covering of $\mathcal{B}$ (proper means with non trivial group). This definition is equivalent to the original one (see [20] for the triangular case and [19, Prop. 4.1] for the non-triangular case) which was introduced in [1]: $\mathcal{B}$ is simply connected if and only if $\pi_1(Q_\mathcal{B}, I) = 1$ for any admissible presentation $kQ_\mathcal{B}/I \simeq \mathcal{B}$ of $\mathcal{B}$ (see [20] for the definition of $\pi_1(Q_\mathcal{B}, I)$).

Basic notions on covering techniques (see [9] and [22]). Let $F \colon \mathcal{E} \to \mathcal{B}$ be a Galois covering with group G. The G-action on $\mathcal{E}$ gives rise to an action of G on $MOD(\mathcal{E})$: if $M \in MOD(\mathcal{E})$ and $g \in G$, then ${}^g M := F \circ g^{-1} \in MOD(\mathcal{E})$. Moreover, F defines two additive functors $F_\lambda \colon MOD(\mathcal{E}) \to MOD(\mathcal{B})$ (the push-down functor) and $F. \colon MOD(\mathcal{B}) \to MOD(\mathcal{E})$ (the pull-up functor) with the following properties (for more details we refer the reader to [9]):

. $F.M = M \circ F$ for any $M \in MOD(\mathcal{B})$,

. if $M \in MOD(\mathcal{E})$, then $(F_\lambda M)(x) = \bigoplus_{x' \in F^{-1}(x)} M(x')$ for any $x \in Ob(\mathcal{B})$. If $u \in {}_y\mathcal{E}_x$, then the restriction of $(F_\lambda M)(F(u))$ to $M(g.x)$ (for $g.x \in F^{-1}(F(x)) = G.x$) is equal to $M(g.u)\colon M(g.x) \to M(g.y)$,

. F_λ and $F.$ are exact and send projective modules to projective modules,

. $F_\lambda \mathcal{E} \simeq \bigoplus_{g \in G} \mathcal{B}$ and $F.\mathcal{B} \simeq \mathcal{E}$, where $\mathcal{E}$ (resp. $\mathcal{B}$) is the $\mathcal{E}$-module $x \mapsto \bigoplus_{y \in Ob(\mathcal{E})} {}_y\mathcal{E}_x$ (resp. the $\mathcal{B}$-module $x \mapsto \bigoplus_{y \in Ob(\mathcal{B})} {}_y\mathcal{B}_x$),

. $F.F_\lambda = \bigoplus_{g \in G} {}^g Id_{MOD(\mathcal{E})}$

. if $X \in MOD(\mathcal{B})$ verifies $X \simeq F_\lambda Y$ for some $Y \in MOD(\mathcal{E})$, then $F_\lambda F.X \simeq \bigoplus_{g \in G} X$,

. $F_\lambda(mod(\mathcal{E})) \subseteq mod(\mathcal{B})$, $F_\lambda(Mod(\mathcal{E})) \subseteq Mod(\mathcal{B})$, $F.(Mod(\mathcal{B})) \subseteq Mod(\mathcal{E})$,

. $D \circ F. = F. \circ D$ and $D \circ F_{\lambda|mod(\mathcal{E})} \simeq F_\lambda \circ D_{|mod(\mathcal{E})}$ where $D = Hom_k(?, k)$ is the usual duality,

. F_λ is left adjoint to $F.$,

. $D \circ F_\lambda \circ D$ is right adjoint to $F.$ (in particular, there is a functorial isomorphism $Hom_\mathcal{E}(F.M, N) \simeq Hom_\mathcal{B}(M, F_\lambda N)$ for any $M \in MOD(\mathcal{B})$ and any $N \in mod(\mathcal{E})$).

. for any $M, N \in MOD(\mathcal{E})$, the following mappings induced by F_λ are bijective:

$$\bigoplus_{g \in G} Hom_\mathcal{E}({}^g M, N) \to Hom_\mathcal{B}(F_\lambda M, F_\lambda N) \text{ and } \bigoplus_{g \in G} Hom_\mathcal{E}(M, {}^g N) \to Hom_\mathcal{B}(F_\lambda M, F_\lambda N)$$

These properties give the following result which will be used many times in this text:

Lemma 1.2. *If $M \in MOD(\mathcal{E})$ (resp. $M \in MOD(\mathcal{B})$) has finite projective dimension, then so does $F_\lambda M$ (resp. $F.(M)$).*

Let $M \in MOD(\mathcal{E})$, $N \in MOD(\mathcal{B})$ and $j \geqslant 1$. There is an isomorphism of vector spaces:

$$Ext^j_{\mathcal{B}}(F_\lambda M, N) \simeq Ext^j_{\mathcal{E}}(M, F.N)$$

Moreover, if $M \in mod(\mathcal{E})$ then there is an isomorphism of vector spaces:

$$Ext^j_{\mathcal{E}}(F.N, M) \simeq Ext^j_{\mathcal{B}}(N, F_\lambda N)$$

Proof: The first assertion is due to the fact that $F.$ and F_λ are exact and send projective modules to projective modules. For the same reasons, $F.$ and F_λ induce $F.: \mathcal{D}(MOD(\mathcal{B})) \to \mathcal{D}(MOD(\mathcal{E}))$ and $F_\lambda: \mathcal{D}(MOD(\mathcal{E})) \to \mathcal{D}(MOD(\mathcal{B}))$ respectively and the adjunctions $(F_\lambda, F.)$ and $(F., F_\lambda)$ at the level of module categories give rise to adjunctions at the level of derived categories. Since $Ext^j_{\mathcal{E}}(X, Y) = Hom_{\mathcal{D}(MOD(\mathcal{E}))}(Y, X[j])$ we get the announced isomorphisms. ∎

Remark that an isomorphism of k-categories is a particular case of Galois covering. When F is an isomorphism, $F.$ and F_λ have additional properties as shows the following lemma whose proof is a direct consequence of the definition of the push-down and pull-up functors.

Lemma 1.3. *Assume that $F: \mathcal{E} \to \mathcal{B}$ is an isomorphism of k-categories. Then $F.F_\lambda = Id_{MOD(\mathcal{E})}$ and $F_\lambda F. = Id_{MOD(\mathcal{B})}$.*

Modules of the first kind. Let $F: \mathcal{E} \to \mathcal{B}$ be a Galois covering with group G. A $\mathcal{B}$-module M is called of the first kind w.r.t. F if and only if for any indecomposable direct summand N of M there exists $\widehat{N} \in MOD(\mathcal{E})$ such that $N \simeq F_\lambda \widehat{N}$. We will denote by $ind_1(\mathcal{B})$ (resp. $mod_1(\mathcal{B})$) the full subcategory of $ind(\mathcal{B})$ (resp. of $mod(\mathcal{B})$) of modules of the first kind w.r.t. F. Notice the following properties of $ind_1(\mathcal{B})$:

. if $M \in ind_1(\mathcal{B})$ and $N \in MOD(\mathcal{E})$ verify $M \simeq F_\lambda N$, then $N \in ind(\mathcal{E})$,

. if $M \in ind_1(\mathcal{B})$ and $N, N' \in MOD(\mathcal{E})$ verify $M \simeq F_\lambda N \simeq F_\lambda N'$, then there exists $g \in G$ such that $N' \simeq {}^g N$.

If $\mathcal{B}$ is connected and if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$, where the $\mathcal{E}_i$'s are the connected components of $\mathcal{E}$, then an indecomposable $\mathcal{B}$-module M is of the first kind w.r.t. F if and only if it is of the first kind w.r.t. $F_i: \mathcal{E}_i \hookrightarrow \mathcal{E} \to \mathcal{B}$ for any $i \in I$. More precisely, we have the following well know lemma where we keep the established notations.

Lemma 1.4. *Let $M \in ind(\mathcal{B})$. If $\widehat{M} \in ind(\mathcal{E})$ is such that $F_\lambda \widehat{M} \simeq M$, then there is a unique $i \in I$ such that $\widehat{M} \in ind(\mathcal{E}_i)$. In such a case, we have $M \simeq (F_i)_\lambda \widehat{M}$. Moreover, if $j \in I$ then there exists $g \in G$ such that $g(\mathcal{E}_i) = \mathcal{E}_j$, and for any such g we have: ${}^g \widehat{M} \in ind(\mathcal{E}_j)$ and $(F_j)_\lambda {}^g \widehat{M} \simeq M$.*

Throughout this text A will denote a basic and connected finite dimensional k-algebra and n will denote the rank of its Grothendieck group $K_0(A)$.

2 Galois coverings associated with modules of the first kind

Throughout this section we will use the following data:

- $F: \mathcal{C} \to A$ a Galois covering with group G

- $T = T_1 \bigoplus \ldots \bigoplus T_n \in mod(A)$ (with $T_i \in ind(A)$) a basic tilting A-module of the first kind w.r.t. F,

- $\lambda_i: F_\lambda(\widehat{T_i}) \to T_i$ an isomorphism with $\widehat{T_i} \in ind(\mathcal{C})$, for every $i \in \{1, \ldots, n\}$.

Let $B = End_A(T)$. With these data, we wish to:

1. construct a Galois covering $F_{\widehat{T_i}, \lambda_i}$ with group G of B,

2. study the dependence of the equivalence class of $F_{\widehat{T_i}, \lambda_i}$ on the data $\widehat{T_i}$, λ_i and on the choice of F in its equivalence class $[F]$,

3. repeat the construction made at the first step starting from T (viewed as a basic tilting B-module) and the Galois covering $F_{\widehat{T_i}, \lambda_i}$. This will give a Galois covering of $End_B(T)$ which will be compared with F using the isomorphism $\rho_A: A \xrightarrow{\sim} End_B(T)$.

2.1   Construction of the Galois covering $F_{\widehat{T}_i,\lambda_i}$

Let $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ be the following k-category:

- the set of objects is $\{ {}^g\widehat{T}_i \mid g \in G,\ i \in \{1,\ldots,n\}\}$ (${}^g\widehat{T}_i$ and ${}^{g'}\widehat{T}_j$ are considered as different objects if $(i,g) \neq (j,g')$),

- the space of morphisms from ${}^g\widehat{T}_i$ to ${}^h\widehat{T}_j$ is equal to $Hom_{\mathcal{C}}({}^g\widehat{T}_i,\ {}^h\widehat{T}_j)$,

- the composition is induced by the composition of morphisms in $mod(\mathcal{C})$.

Remark 2.1.   *1. The $\mathcal{C}$-modules $\bigoplus_{g,i} {}^g\widehat{T}_i$ and $F.T$ are isomorphic.*

2. If G is a finite group, then $\mathcal{C}$ is a finite dimensional k-algebra. In particular, $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ and $End_{\mathcal{C}}(F.T)$ are isomorphic k-algebras.

3. The G-action on $mod(\mathcal{C})$ naturally endows $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ with a structure of free G-category.

4. $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is locally bounded if and only if $G_{\widehat{T}_i} = 1$ for any i. This is equivalent to say that $F.T$ is a basic $\mathcal{C}$-module.

The isomorphisms $\lambda_1,\ldots,\lambda_n$ define the following functor:

$$
\begin{array}{rccc}
F_{\widehat{T}_i,\lambda_i} : & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i) & \longrightarrow & B \\
& {}^g\widehat{T}_i & \longmapsto & T_i \\
& {}^g\widehat{T}_i \xrightarrow{u} {}^h\widehat{T}_j & \longmapsto & T_i \xrightarrow{\ \lambda_j\ \ F_\lambda u\ \ \lambda_i^{-1}\ } T_j
\end{array}
$$

Lemma 2.2. *The functor $F_{\widehat{T}_i,\lambda_i} : \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i) \to B$ is a Galois covering with group G.*

Proof: For simplicity, we shall write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ and $F' : \mathcal{C}' \to B$ for $F_{\widehat{T}_i,\lambda_i}$. Recall (see Remark 2.1) that G acts freely on $\mathcal{C}'$. Moreover, we have $F' \circ g = F'$ by construction of F'. So, F' defines a commutative diagram of k-categories and k-linear functors:

$$
\begin{array}{ccc}
\mathcal{C}' & & \\
\downarrow & \searrow{\scriptstyle F'} & \\
\mathcal{C}'/G & \xrightarrow[\overline{F'}]{} & B
\end{array}
\qquad (\star)
$$

Where $\mathcal{C}' \to \mathcal{C}'/G$ is the quotient functor. From the properties verified by F_λ (see Section 1) we infer that F' is a covering functor. Since $\mathcal{C}' \to \mathcal{C}'/G$ is a also covering functor we deduce that so is $\overline{F'}$ (see Lemma 1.1). Finally, $\overline{F'}$ restricts to a bijective mapping $Ob(\mathcal{C}')/G = \{ {}^g\widehat{T}_i \mid g \in G,\ i \in \{1,\ldots,n\}\}/G \to Ob(B) = \{T_1,\ldots,T_n\}$ so $\overline{F'}$ is an isomorphism. Thus, F' is a Galois covering with group G. $\blacksquare$

Since $F_{\widehat{T}_i,\lambda_i}$ is a Galois covering, it is natural to ask whether $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is connected or not. The following lemma partially answers this question.

Lemma 2.3. *If $\mathcal{C}$ is not connected, then $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is not connected.*

Proof: For simplicity let us write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$. Assume that $\mathcal{C}$ is not connected and let $\mathcal{C} = \coprod_{x \in I} \mathcal{C}_x$ where the $\mathcal{C}_x$'s are the connected components of $\mathcal{C}$. For $i \in \{1,\ldots,n\}$, we have $\widehat{T}_i \in ind(\mathcal{C})$, so there exists a unique $x_i \in I$ such that $\widehat{T}_i \in ind(\mathcal{C}_{x_i})$. Let us set:

$$
G_{x_1} = \{g \in G \mid g(\mathcal{C}_{x_1}) = \mathcal{C}_{x_1}\}
$$

Let $i \in \{1,\ldots,n\}$, since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$, there exists $g_i \in G$ such that $g_i(\mathcal{C}_{x_1}) = \mathcal{C}_{x_i}$ (in particular $g_1 \in G_{x_1}$). Therefore:

$$
(\forall i \in \{1,\ldots,n\})\ \ {}^{g_i^{-1}}\widehat{T}_i \in mod(\mathcal{C}_{x_1})
$$

Let us set O to be the following set of objects of $\mathcal{C}'$:

$$
O := \{ {}^g\widehat{T}_i \mid i \in \{1,\ldots,n\} \text{ and } gg_i \in G_{x_1}\} \subseteq Ob(\mathcal{C}')
$$

Remark that O satisfies the following:

. $O \neq \emptyset$ because $\widehat{T}_1 \in O$.

. Since $\mathcal{C}$ is not connected and since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$ we have $G_{x_1} \subsetneq G$. Let $g \in G\backslash G_{x_1}$, then $gg_1 \notin G_{x_1}$ and ${}^g\widehat{T}_1 \notin O$. Hence $O \subsetneq Ob(\mathcal{C})$.

. For any ${}^g\widehat{T}_i \in Ob(\mathcal{C}')$, we have ${}^g\widehat{T}_i \in O$ if and only if ${}^g\widehat{T}_i \in ind(\mathcal{C}_{x_1})$. As a consequence, there is no non zero morphism in $\mathcal{C}'$ between an object in O and an object in $Ob(\mathcal{C}')\backslash O$.

As a consequence, $\mathcal{C}'$ is not connected. $\blacksquare$

2.2 Independence of the equivalence class of $F_{\widehat{T}_i, \lambda_i}$ on the data $F, \widehat{T}_i, \lambda_i$

In the two following lemmas, we examine the dependence of the equivalence class $[F_{\widehat{T}_i, \lambda_i}]$ of $F_{\widehat{T}_i, \lambda_i}$ on the choice of $\widehat{T}_1, \ldots, \widehat{T}_n, \lambda_1, \ldots, \lambda_n$ and on the choice of F in its equivalence class $[F]$.

Lemma 2.4. *For each $i \in \{1, \ldots, n\}$, let $\mu_i \colon F_\lambda \overline{T}_i \to T_i$ be an isomorphism with $\overline{T}_i \in ind(\mathcal{C})$. Then $F_{\widehat{T}_i, \lambda_i}$ and $F_{\overline{T}_i, \mu_i}$ are equivalent.*

Proof: We need to exhibit a commutative square:

$$
\begin{array}{ccc}
\mathcal{E}nd_\mathcal{C}(\bigoplus_{i,g} {}^g\overline{T}_i) & \xrightarrow{\quad \varphi \quad} & \mathcal{E}nd_\mathcal{C}(\bigoplus_{i,g} {}^g\widehat{T}_i) \\
{\scriptstyle F_{\overline{T}_i, \mu_i}} \downarrow & & \downarrow {\scriptstyle F_{\widehat{T}_i, \lambda_i}} \\
B & \xrightarrow{\quad \psi \quad} & B
\end{array}
\qquad (\star)
$$

where φ, ψ are isomorphisms and where $\psi(x) = x$ for any $x \in Ob(B) = \{T_1, \ldots, T_n\}$. Let $i \in \{1, \ldots, n\}$. We have $F_\lambda \overline{T}_i \simeq T_i \simeq F_\lambda \widehat{T}_i$, so there exists an isomorphism $\theta_i \colon \overline{T}_i \xrightarrow{\sim} {}^{g_i}\widehat{T}_i$ with $g_i \in G$. Let us define φ by:

$$
\begin{array}{cccc}
\varphi : & \mathcal{E}nd_\mathcal{C}(\bigoplus_{i,g} {}^g\overline{T}_i) & \longrightarrow & \mathcal{E}nd_\mathcal{C}(\bigoplus_{i,g} {}^g\widehat{T}_i) \\
& {}^g\overline{T}_i & \longmapsto & {}^{gg_i}\widehat{T}_i \\
& {}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j & \longmapsto & {}^{gg_i}\widehat{T}_i \xrightarrow{{}^h\theta_j \, u \, {}^g\theta_i^{-1}} {}^{hg_j}\widehat{T}_j
\end{array}
$$

Then φ is an isomorphism of k-categories. Notice that θ_i defines an isomorphism $F_\lambda \theta_i \colon F_\lambda \overline{T}_i \to F_\lambda \widehat{T}_i$. So we can define ψ by:

$$
\begin{array}{cccc}
\psi : & B & \longrightarrow & B \\
& T_i & \longmapsto & T_i \\
& T_i \xrightarrow{u} T_j & \longmapsto & \psi(u)
\end{array}
$$

where $\psi(u)$ is the composition:

$$
T_i \xrightarrow{\lambda_i^{-1}} F_\lambda \widehat{T}_i \xrightarrow{F_\lambda \theta_i^{-1}} F_\lambda \overline{T}_i \xrightarrow{\mu_i} T_i \xrightarrow{u} T_j \xrightarrow{\mu_j^{-1}} F_\lambda \overline{T}_j \xrightarrow{F_\lambda \theta_j} F_\lambda \widehat{T}_j \xrightarrow{\lambda_j} T_j
$$

So ψ is an isomorphism of k-categories which restricts to the identity map on $Ob(B)$. Moreover φ and ψ make $(\star)$ commutative. $\blacksquare$

In the following lemma, we show that, under additional hypotheses on T, the equivalence class of $F_{\widehat{T}_i, \lambda_i}$ does not depend on the choice of F in $[F]$.

Lemma 2.5. *Assume that $F' \colon \mathcal{C}' \to A$ is a Galois covering (with group G) equivalent to F and assume that T verifies the following condition:*

$(H_{A,T})$:"$\psi.T_i \simeq T_i$ for any i and for any isomorphism $\psi \colon A \xrightarrow{\sim} A$ which restricts to the identity map on $Ob(A)$."

Then T is of the first kind w.r.t. F'. For each $i \in \{1, \ldots, n\}$ let $\mu_i \colon F'_\lambda \overline{T}_i \to T_i$ be an isomorphism with $\overline{T}_i \in ind(\mathcal{C}')$. Then $F'_{\overline{T}_i, \mu_i}$ and $F_{\widehat{T}_i, \lambda_i}$ are equivalent.

Proof: Let us fix an isomorphism between F and F':

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\varphi} & \mathcal{C} \\
{\scriptstyle F'} \downarrow & & \downarrow {\scriptstyle F} \\
A & \xrightarrow{\psi} & A
\end{array}
$$

Let us set $\nu\colon Aut(\mathcal{C}') \to Aut(\mathcal{C})$ to be the isomorphism of groups (recall that $Aut(\mathcal{C}') = G$ and $Aut(\mathcal{C}) = G$):

$$\nu\colon \quad Aut(\mathcal{C}') \quad \to \quad Aut(\mathcal{C})$$
$$g \quad \mapsto \quad \varphi \circ g \circ \varphi^{-1}$$

Recall that any $g \in Aut(\mathcal{C}) = G$ (resp. $g \in Aut(\mathcal{C}') = G$) defines an automorphism g of $MOD(\mathcal{C})$ (resp. of $MOD(\mathcal{C}')$). Therefore we have an equality of functors $MOD(\mathcal{C}') \to MOD(A)$:

$$(\forall g \in Aut(\mathcal{C}')) \quad \varphi_\lambda \circ g = \nu(g) \circ \varphi_\lambda$$

Let us fix an isomorphism $\theta_i \colon \psi.T_i \to T_i$, for each i, and let us set $\overline{T}_i = \varphi.\widehat{T}_i$. In particular: $\varphi_\lambda \overline{T}_i = \widehat{T}_i$ (see Lemma 1.3). Since $\psi.\psi_\lambda = Id_{MOD(A)}$ (loc. cit.) and $\psi F' = F\varphi$, we infer that:

$$F'_\lambda \overline{T}_i = \psi.\psi_\lambda F'_\lambda \overline{T}_i = \psi.F_\lambda \varphi_\lambda \overline{T}_i = \psi.F_\lambda \widehat{T}_i$$

Therefore, we get for each i an isomorphism $\mu_i \colon F'_\lambda \overline{T}_i \to T_i$ equal to the composition:

$$\mu_i\colon F'_\lambda \overline{T}_i = \psi.F_\lambda \widehat{T}_i \xrightarrow{\psi.\lambda_i} \psi.T_i \xrightarrow{\theta_i} T_i$$

This proves that T is of the first kind w.r.t. F'. According to the preceding subsection, this defines the Galois covering with group G:

$$F'_{\overline{T}_i,\mu_i} \colon \mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{g,i} {}^g\overline{T}_i) \to B$$

Thanks to Lemma 2.4 we only need to prove that $F'_{\overline{T}_i,\mu_i}$ and $F_{\widehat{T}_i,\lambda_i}$ are equivalent.

First, we have the following functor induced by φ_λ:

$$\overline{\varphi}\colon \quad \mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{i,g} {}^g\overline{T}_i) \quad \longrightarrow \quad \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$$
$$ {}^g\overline{T}_i \quad \longmapsto \quad {}^{\nu(g)}\widehat{T}_i = \varphi_\lambda {}^g\overline{T}_i$$
$$ {}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j \quad \longmapsto \quad {}^{\nu(g)}\widehat{T}_i \xrightarrow{\varphi_\lambda u} {}^{\nu(h)}\widehat{T}_j$$

Since $\nu\colon G \to G$ is an isomorphism and because of the equalities $\varphi_\lambda \varphi. = Id_{MOD(\mathcal{C})}$ and $\varphi.\varphi_\lambda = Id_{MOD(\mathcal{C})}$ (see Lemma 1.3), the functor $\overline{\varphi}$ is an isomorphism.

Secondly, we have the following functor induced by ψ_λ:

$$\overline{\psi}\colon \quad B \quad \longrightarrow \quad B$$
$$T_i \quad \longmapsto \quad T_i$$
$$T_i \xrightarrow{u} T_j \quad \longmapsto \quad T_i \xrightarrow{\psi_\lambda(\theta_j^{-1}u\theta_i)} T_j$$

Since $\psi_\lambda \psi. = \psi.\psi_\lambda = Id_{MOD(A)}$, the functor $\overline{\psi}$ is a well defined isomorphism and restricts to the identity map on $Ob(B)$. Therefore, we have a diagram whose horizontal arrows are isomorphisms and whose bottom horizontal arrow restricts to the identity map on the set of objects:

$$
\begin{array}{ccc}
\mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{i,g} {}^g\overline{T}_i) & \xrightarrow{\overline{\varphi}} & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i) \\
{\scriptstyle F'_{\overline{T}_i,\mu_i}}\downarrow & & \downarrow {\scriptstyle F_{\widehat{T}_i,\lambda_i}} \\
B & \xrightarrow{\overline{\psi}} & B
\end{array}
$$

This diagram is commutative, indeed, for any ${}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j$ we have:

$$
\begin{aligned}
\overline{\psi} F'_{\overline{T}_i,\mu_i}(u) &= \overline{\psi}(\mu_j F'_\lambda(u)\mu_i^{-1}) = \psi_\lambda(\theta_j^{-1}\mu_j F'_\lambda(u)\mu_i^{-1}\theta_i) \\
&= \psi_\lambda(\theta_j^{-1}\theta_j\psi.(\lambda_j)F'_\lambda(u)\psi.(\lambda_i)^{-1}\theta_i^{-1}\theta_i) \\
&= \lambda_j \, (\psi_\lambda F'_\lambda)(u) \, \lambda_i^{-1} && \text{because } \psi_\lambda\psi. = Id_{MOD(\mathcal{C})} \\
&= \lambda_j \, (F_\lambda \varphi_\lambda)(u) \, \lambda_i^{-1} && \text{because } F\varphi = \psi F' \\
&= F_{\widehat{T}_i,\lambda_i}(\varphi_\lambda(u)) = F_{\widehat{T}_i,\lambda_i}\overline{\varphi}(u)
\end{aligned}
$$

This proves that $F'_{\overline{T}_i,\mu_i}$ and $F_{\widehat{T}_i,\lambda_i}$ are equivalent. ∎

Later, we shall prove that if T is a basic tilting A-module lying in the connected component of $\vec{\mathcal{K}}_A$ containing A, then the hypothesis $(H_{A,T})$ in the preceding lemma is automatically verified. As a consequence, for these tilting A-modules, the property "to be of the first kind w.r.t. an equivalence class of Galois coverings of A" does make sense. However, the hypothesis $(H_{A,T})$ is not verified for any A-module T as the following example shows.

Example 2.6. *Let A be the path algebra of the following quiver:*
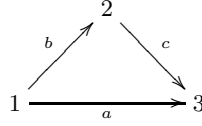
Here $n = 3$ and we have an isomorphism of k-categories: $\psi\colon A \xrightarrow{\sim} A$ such that $\psi(x) = x$ for any $x \in Ob(A)$, $\psi(a) = a + cb$, $\psi(b) = b$ and $\psi(c) = c$. For any integer i, let T_i be the following A-module:
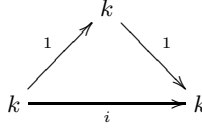
Then:

- *T_i and T_{i+1} are not isomorphic, for any i,*
- *if $car(k) \neq 2$, then T_1, T_2, T_3 are pairwise non isomorphic,*
- *$\psi.T_i = T_{i+1}$ for any i.*

In particular, if $car(k) \neq 2$, then hypothesis $(H_{A,T})$ is not satisfied for $T = T_1 \oplus T_2 \oplus T_3$. Remark that T is not tilting. Indeed, for any i, we have $Ext_A^1(T_i, T_i) \simeq k$ because $\tau_A(T_i) \simeq T_i$.

Remark 2.1, Lemma 2.4 and Lemma 2.5 justify the following definition:

Definition 2.7. *Assume that the hypothesis $(H_{A,T})$ is satisfied (see Lemma 2.5). The equivalence class $[F]$ of F and the basic tilting A-module $T = T_1 \bigoplus \ldots \bigoplus T_n$ of the first kind w.r.t. $[F]$ (with $T_i \in ind(A)$) uniquely define an equivalence class of Galois covering of B with group G and which admits $F_{\widehat{T_i}, \lambda_i}$ as a representative. This equivalence class will be denoted by $[F]_T\colon \mathcal{E}nd_{\mathcal{C}}(F.T) \to B$.*

2.3 Comparison of $[F]$ and $([F]_T)_T$

For short, let us write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} \, {}^g\widehat{T_i})$ and F' for $F_{\widehat{T_i}, \lambda_i}$. In this subsection we shall not assume that the hypothesis $(H_{A,T})$ of Lemma 2.5 is satisfied, except for the last proposition. Starting from F and from the isomorphisms $\lambda_i\colon F_\lambda \widehat{T_i} \xrightarrow{\sim} T_i$, $i \in \{1, \ldots, n\}$, we have constructed the Galois covering F' of B. One may try to perform the same construction starting from F' in order to get a Galois covering F'' of $End_B(T) \simeq A$ and eventually compare F'' with F. In this purpose, we need to prove that T is of the first kind w.r.t. F'. Let us fix a lifting $L\colon Ob(A) \to Ob(\mathcal{C})$ of the surjective mapping $F\colon Ob(\mathcal{C}) \to Ob(A)$. For $x \in Ob(A)$, let $\widehat{T(x)}$ be the $\mathcal{C}'$-module such that:

- $\widehat{T(x)}({}^g\widehat{T_i}) = \widehat{T_i}(g^{-1}L(x))$ for any ${}^g\widehat{T_i} \in Ob(\mathcal{C}')$,
- $\widehat{T(x)}\left({}^g\widehat{T_i} \xrightarrow{u} {}^h\widehat{T_j}\right)$ is equal to $\widehat{T_i}(g^{-1}L(x)) \xrightarrow{u_{L(x)}} \widehat{T_j}(h^{-1}L(x))$ for any $u \in {}_{h\widehat{T_j}}\mathcal{C}'_{g\widehat{T_i}}$.

Therefore, for any $i \in \{1, \ldots, n\}$, we have:

$$\left(F'_\lambda \widehat{T(x)}\right)(T_i) = \bigoplus_{g \in G} \widehat{T(x)}({}^g\widehat{T_i}) = \bigoplus_{g \in G} \widehat{T_i}(g^{-1}L(x)) = \left(F_\lambda \widehat{T_i}\right)(x)$$

So we may set $(\mu_x)_{T_i}\colon \left(F'_\lambda \widehat{T(x)}\right)(T_i) \to (T(x))(T_i)$ to be equal to $(F_\lambda \widehat{T_i})(x) \xrightarrow{(\lambda_i)_x} T_i(x)$.

Lemma 2.8. *The linear isomorphisms $(\mu_x)_{T_i}$ $(i \in \{1, \ldots, n\})$ define an isomorphism of B-modules:*

$$\mu_x\colon F'_\lambda \widehat{T(x)} \xrightarrow{\sim} T(x)$$

Proof: We only need to prove that μ_x is a morphism of B-modules. Let $u \in {}_t\widehat{T_j}\mathcal{C}'_s\widehat{T_i}$ so that $F'(u) \in {}_{T_j}B_{T_i}$, and let us prove that the following diagram commutes:

$$(\star)$$

$$
\begin{array}{ccc}
\left(F'_\lambda \widehat{T(x)}\right)(T_i) & \xrightarrow{(\mu_x)_{T_i} = (\lambda_i)_x} & T_i(x) \\
{\scriptstyle \left(F'_\lambda \widehat{T(x)}\right)(F'(u))} \Big\downarrow & & \Big\downarrow {\scriptstyle (T(x))(F'(u)) = F'(u)_x} \\
\left(F'_\lambda \widehat{T(x)}\right)(T_j) & \xrightarrow{(\mu_x)_{T_j} = (\lambda_j)_x} & T_j(x)
\end{array}
$$

Let $g \in G$ and let us compute the restriction of $F'(u)_x \circ (\lambda_i)_x$ to $\widehat{T(x)}(\,^g\widehat{T}_i)$. Recall that $F'(u)_x$ is equal to the composition:

$$T_i(x) \xrightarrow{(\lambda_i^{-1})_x} \left(F_\lambda\,^s\widehat{T}_i\right)(x) \xrightarrow{(F_\lambda u)_x} \left(F_\lambda\,^t\widehat{T}_j\right)(x) \xrightarrow{(\lambda_j)_x} T_j(x)$$

Moreover, the restriction of $(F_\lambda u)_x$ to $\widehat{T(x)}(\,^g\widehat{T}_i) = \widehat{T}_i(g^{-1}L((x))$ is (by construction of the push-down functor) equal to $\widehat{T}_i(g^{-1}L(x)) \xrightarrow{u_{sg^{-1}L(x)}} \widehat{T}_j(t^{-1}sg^{-1}L(x))$. Thus, the restriction of $F'(u)_x \circ (\lambda_i)_x$ to $\widehat{T(x)}(\,^g\widehat{T}_i)$ is equal to the composition:

$$\widehat{T}_i(g^{-1}L(x)) \xrightarrow{u_{sg^{-1}L(x)}} \widehat{T}_j(t^{-1}sg^{-1}L(x)) \xrightarrow{(\lambda_j)_x} T_j(x) \qquad (i)$$

On the other hand, the restriction of $\left(F'_\lambda\widehat{T(x)}\right)(F'(u))$ to $\widehat{T(x)}(\,^g\widehat{T}_i) = \widehat{T(x)}(\,^{gs^{-1}s}\widehat{T}_i)$ is (by construction of the push-down functor) equal to:

$$\widehat{T(x)}(\,^g\widehat{T}_i) \xrightarrow{\widehat{T(x)}(\,^{gs^{-1}}u)} \widehat{T(x)}(\,^{gs^{-1}t}\widehat{T}_j) \qquad (ii)$$

and $\widehat{T(x)}(\,^{gs^{-1}}u) = \left(\,^{gs^{-1}}u\right)_{L(x)} = u_{sg^{-1}L(x)}$. This last equality, together with (i) and (ii), proves that the diagram $(\star)$ commutes. $\blacksquare$

Thanks to Lemma 2.8, we have a Galois covering $F'_{\widehat{T(x)},\mu_x}: \mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{g,x}\,^g\widehat{T(x)}) \to End_B(T)$ with group G. For short, we shall write $\mathcal{C}''$ for $\mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{g,x}\,^g\widehat{T(x)})$ and F'' for $F'_{\widehat{T(x)},\mu_x}$. The following lemma relates F'' and F.

Lemma 2.9. *There exists an isomorphism of k-categories $\psi: \mathcal{C} \xrightarrow{\sim} \mathcal{C}''$ such that the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \psi\ } & \mathcal{C}'' \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F''} \\
A & \xrightarrow{\ \rho_A\ } & End_B(T)
\end{array}
$$

In particular, F and $\rho_A^{-1}F''$ are equivalent as Galois coverings of A.

Proof: Since G acts freely on $Ob(\mathcal{C})$ and since $L: Ob(A) \to Ob(\mathcal{C})$ lifts $F: Ob(\mathcal{C}) \to Ob(A)$, any $x \in Ob(\mathcal{C})$ is equal to $gL(x')$ with $g \in G, x' \in Ob(A)$ uniquely determined by x. Let $\psi: \mathcal{C} \to \mathcal{C}''$ be as follows:

- $\psi(gL(x)) = \,^g\widehat{T(x)}$ for any $gL(x) \in Ob(\mathcal{C})$,

- for $u \in \,_{hL(y)}\mathcal{C}_{gL(x)}$, we let $\psi(u): \,^g\widehat{T(x)} \to \,^h\widehat{T(y)}$ be the morphism of $\mathcal{C}'$-modules such that for any $\,^s\widehat{T}_i \in Ob(\mathcal{C}')$, $\psi(u)_{\,^s\widehat{T}_i}$ is equal to:

$$\widehat{T}_i(s^{-1}u): \widehat{T}_i(s^{-1}gL(x)) \to \widehat{T}_i(s^{-1}hL(y))$$

Let us prove the following facts:

1. $\psi(u)$ is a morphism of $\mathcal{C}'$-modules for any $u \in \,_{hL(y)}\mathcal{C}_{gL(x)}$,

2. ψ is a functor,

3. $F'' \circ \psi = \rho_A \circ F$,

4. ψ is an isomorphism.

1) Let $u \in \,_{hL(y)}\mathcal{C}_{gL(x)}$. We need to prove that for any $f \in \,_{t\widehat{T}_j}\mathcal{C}'_{s\widehat{T}_i}$, the following diagram commutes:

$$
\begin{array}{ccc}
\,^g\widehat{T(x)}(\,^s\widehat{T}_i) & \xrightarrow{\psi(u)_{\,^s\widehat{T}_i}} & \,^h\widehat{T(y)}(\,^s\widehat{T}_i) \\
{\scriptstyle \,^g\widehat{T(x)}(f)}\downarrow & & \downarrow{\scriptstyle \,^h\widehat{T(y)}(f)} \\
\,^g\widehat{T(x)}(\,^t\widehat{T}_j) & \xrightarrow{\psi(u)_{\,^t\widehat{T}_j}} & \,^h\widehat{T(y)}(\,^t\widehat{T}_j)
\end{array}
$$

By construction, this diagram is equal to:

$$\begin{array}{ccc}
\widehat{T}_i(s^{-1}gL(x)) & \xrightarrow{\widehat{T}_i(s^{-1}u)} & \widehat{T}_i(s^{-1}hL(y)) \\
{\scriptstyle f_{gL(x)}} \downarrow & & \downarrow {\scriptstyle f_{hL(y)}} \\
\widehat{T}_j(t^{-1}gL(x)) & \xrightarrow{\widehat{T}_j(t^{-1}u)} & \widehat{T}_j(t^{-1}hL(y))
\end{array}$$

and the latter is commutative because $f\colon {}^s\widehat{T}_i \to {}^t\widehat{T}_j$ is a morphism of $\mathcal{C}$-modules. This proves that $\psi(u)$ is a morphism of $\mathcal{C}'$-modules for any morphism u in $\mathcal{C}$.

2) One easily checks that $\psi(1_{gL(x)}) = Id_{g\widehat{T(x)}}$ for any $gL(x) \in Ob(\mathcal{C})$. Let u, v be morphisms in $\mathcal{C}$ such that the composition vu exists. Then, for any ${}^s\widehat{T}_j \in Ob(\mathcal{C}')$:

$$(\psi(v) \circ \psi(u))_{s\widehat{T}_j} = \psi(v)_{s\widehat{T}_j} \circ \psi(u)_{s\widehat{T}_j} = \widehat{T}_j(s^{-1}v) \circ \widehat{T}_j(s^{-1}u) = \widehat{T}_j(s^{-1}(v \circ u)) = \psi(v \circ u)_{s\widehat{T}_j}$$

So $\psi\colon \mathcal{C} \to \mathcal{C}''$ is a functor.

3) Let $gL(x) \in Ob(\mathcal{C})$. Then:

$$F'' \circ \psi(gL(x)) = F''({}^g\widehat{T(x)}) = T(x) = \rho_A(x) = \rho_A \circ F(gL(x))$$

Let $u \in {}_{hL(y)}\mathcal{C}_{gL(x)}$ and let us prove that $F''\psi(u) = \rho_A F(u)$. Let $T_i \in Ob(B)$. Then:

$$\begin{cases}
(F''\psi(u))_{T_i} & = \quad T_i(x) \xrightarrow{(\lambda_i^{-1})_x} \left(F'_\lambda \, {}^g\widehat{T(x)}\right)(T_i) \xrightarrow{\left(F'_\lambda(\psi(u))\right)_{T_i}} \left(F'_\lambda \, {}^h\widehat{T(y)}\right)(T_i) \xrightarrow{(\lambda_i)_y} T_i(y) \\
(\rho_A F(u))_{T_i} & = \quad T_i(x) \xrightarrow{T_i(F(u))} T_i(y)
\end{cases}$$

Recall that $\left(F'_\lambda \, {}^g\widehat{T(x)}\right)(T_i) = \bigoplus_{s \in G} {}^g\widehat{T(x)}({}^s\widehat{T}_i)$ and that ${}^g\widehat{T(x)}({}^s\widehat{T}_i) = \widehat{T}_i(s^{-1}gL(x))$, for any $s \in G$. Let $s \in G$. Then, the restriction of $(F'_\lambda(\psi(u)))_{T_i}$ to ${}^g\widehat{T(x)}({}^s\widehat{T}_i)$ is equal to:

$$\widehat{T}_i(s^{-1}gL(x)) \xrightarrow{\psi(u)_{s\widehat{T}_i} = \widehat{T}_i(s^{-1}u)} \widehat{T}_i(s^{-1}hL(y))$$

Therefore, for any $s \in G$, the restriction of $(\lambda_i^{-1})_y \circ (F''\psi(u))_{T_i} \circ (\lambda_i)_x$ (resp. $(\lambda_i^{-1})_y \circ (\rho_A F(u))_{T_i} \circ (\lambda_i)_x$) to ${}^g\widehat{T(x)}({}^s\widehat{T}_i) = \widehat{T}_i(s^{-1}gL(x))$ is equal to:

$$\widehat{T}_i(s^{-1}gL(x)) \xrightarrow{\widehat{T}_i(s^{-1}u)} \widehat{T}_i(s^{-1}hL(y))$$

$$\text{resp.} \quad \widehat{T}_i(s^{-1}gL(x)) \xrightarrow{(\lambda_i^{-1})_y \, T_i(F(u)) \, (\lambda_i)_x} \widehat{T}_i(s^{-1}hL(y))$$

Since $\lambda_i\colon F_\lambda\left({}^s\widehat{T}_i\right) \to T_i$ is an isomorphism of A-modules, $(\lambda_i^{-1})_y \circ T_i(F(u)) \circ (\lambda_i)_x$ equals $\widehat{T}_i(s^{-1}u)$. We infer that $(\lambda_i^{-1})_y \circ (F''\psi(u))_{T_i} \circ (\lambda_i)_x$ and $(\lambda_i^{-1})_y \circ (\rho_A F(u))_{T_i} \circ (\lambda_i)_x$ coincide on $\widehat{T}(s^{-1}gL(x))$, for any $s \in G$. As a consequence, $(F''\psi(u))_{T_i} = (\rho_A F(u))_{T_i}$, for any $T_i \in Ob(B)$. This proves that $F'' \circ \psi(u) = \rho_A \circ F(u)$ for any morphism u in $\mathcal{C}$. In other words: $F'' \circ \psi = \rho_A \circ F$.

4) Let us prove that ψ is an isomorphism. Since F'' and $\rho_A \circ F$ are covering functors, Lemma 1.1 implies that so does ψ. Since ψ restricts to a bijective mapping on objects, we deduce that ψ is an isomorphism. ∎

Thanks to Lemma 2.9 we can complete Lemma 2.3 concerning the connectedness of $\mathcal{C}'$. The following proposition will be useful in the sequel, it is a direct consequence of Lemma 2.3 and Lemma 2.9.

Proposition 2.10. *$\mathcal{C}$ is connected if and only if $\mathcal{C}' = \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} \widehat{T}_i)$ is connected.*

We finish this subsection with the following proposition which compares the equivalence class of F and $([F]_T)_T$ when the latter is well defined (see Definition 2.7). It is a direct consequence of Lemma 2.9. Notice that $\rho_A^{-1} \circ ([F]_T)_T$ is an equivalence class of Galois coverings of A.

Proposition 2.11. *Assume that both conditions $(H_{A,T})$ and $(H_{B,T})$ are satisfied. Then, the equivalence class $[F]$ of F coincides with $\rho_A^{-1} \circ ([F]_T)_T$.*

3 Tilting modules of the first kind

Let $F: \mathcal{C} \to A$ be a Galois covering with group G and with $\mathcal{C}$ locally bounded. The aim of this section is to give "simple" sufficient conditions which guarantee the following facts:

- T is of the first kind w.r.t. F,
- $F.T$ is a basic $\mathcal{C}$-module,
- the hypothesis $(H_{A,T})$ is satisfied (see Lemma 2.5), i.e. $\psi.N \simeq N$ for any direct summand N of T and for any automorphism $\psi: A \xrightarrow{\sim} A$ which restricts to the identity map on objects.

We begin with the following proposition.

Proposition 3.1. *Assume that T and T' lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$. Then:*

$$T \in mod_1(A) \Leftrightarrow T' \in mod_1(A)$$

In particular, if $T' = A$ or $T' = DA$, then $T \in mod_1(A)$.

Proof: Since $A, DA \in mod_1(A)$, we only need to prove the equivalence of the proposition under the assumption: there is an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_A$. Let us assume that $T \in mod_1(A)$. Since $T \to T'$ is an arrow in $\overrightarrow{\mathcal{K}}_A$, we have the following data:

. $T = X \bigoplus \overline{T}$ with $X \in ind(A)$,
. $T' = Y \bigoplus \overline{T}$ with $Y \in ind(A)$,
. $\varepsilon: 0 \to X \to M \to Y \to 0$ a non split exact sequence in $mod(A)$ with $M \in add(\overline{T})$.

Thus, we only need to prove that $Y \in mod_1(A)$ in order to get $T' \in mod_1(A)$. In this purpose, we will need the following lemma.

Lemma 3.2. *Let $\varepsilon : 0 \to X \xrightarrow{u} M \to Y \to 0$ be an exact sequence in $mod(A)$ verifying the following hypotheses:*

. *$X, Y \in ind(A)$ and $X = F_\lambda \widehat{X}$ (with $\widehat{X} \in ind(\mathcal{C})$),*
. *$M = M_1 \bigoplus \ldots \bigoplus M_t$ where $M_i = F_\lambda \widehat{M_i} \in ind(A)$ (with $\widehat{M_i} \in ind(\mathcal{C})$), for every i,*
. *$Ext_A^1(Y, M) = 0$.*

Then (ε) is isomorphic to an exact sequence in $mod(A)$:

$$0 \to X \xrightarrow{\begin{bmatrix} F_\lambda u_1' \\ \vdots \\ F_\lambda u_t' \end{bmatrix}} M_1 \bigoplus \ldots \bigoplus M_T \to Y \to 0$$

where $u_i' \in Hom_\mathcal{C}(\widehat{X}, {}^{g_i}\widehat{M_i})$ for some $g_i \in G$, for every i.

Proof of Lemma 3.2: For short, we shall say that $u \in Hom_A(X, M_i)$ is homogeneous of degree $g \in G$ if and only if $u = F_\lambda u'$ with $u' \in Hom_\mathcal{C}(\widehat{X}, {}^g\widehat{M_i})$. Recall from Section 1 that any $u \in Hom_A(X, M_i)$ is (uniquely) the sum of d homogeneous morphisms with pairwise different degrees (with $d \geqslant 0$). Let us write $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$ with $u_i: X \to M_i$ for each i. We may assume that $u_1: X \to M_1$ is not homogeneous. Thus:

$$u_1 = h_1 + \ldots + h_d$$

where $d \geqslant 2$ and $h_1, \ldots, h_d: X \to M_1$ are non zero homogeneous morphisms of pairwise different degree. In order to prove the lemma, it suffices to prove the following property which we denote by $(\mathcal{P})$:

" *(ε) is isomorphic to an exact sequence of the form:*

$$0 \to X \xrightarrow{\begin{bmatrix} u_1' \\ u_2 \\ \vdots \\ u_t \end{bmatrix}} M_1 \bigoplus \ldots \bigoplus M_t \to Y \to 0 \qquad (\varepsilon')$$

where u_1' is the sum of at most $d - 1$ non zero homogeneous morphisms of pairwise different degree. "

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For simplicity we adopt the following notations:

. $\overline{M} = M_2 \bigoplus \cdots \bigoplus M_t$ (so $M = M_1 \bigoplus \overline{M}$),

. $\overline{u} = \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix} : X \to \overline{M}$ (so $u = \begin{bmatrix} u_1 \\ \hline \overline{u} \end{bmatrix} : X \to M_1 \bigoplus \overline{M}$),

. $\overline{h} = h_2 + \ldots + h_d : X \to M_1$ (so $u_1 = h_1 + \overline{h}$).

From $Hom_A(\varepsilon, M_1)$ we get the exact sequence:

$$Hom_A(M_1 \bigoplus \overline{M}, M_1) \xrightarrow{u^*} Hom_A(X, M_1) \to Ext_A^1(Y, M_1) = 0$$

So there exists $[\lambda, \mu] : M_1 \bigoplus \overline{M} \to M_1$ such that $h_1 = [\lambda, \mu]u$. Hence:

$$h_1 = \lambda u_1 + \mu\overline{u} = \lambda h_1 + \lambda\overline{h} + \mu\overline{u} \qquad (i)$$

Let us distinguish two cases whether $\lambda \in End_A(M_1)$ is invertible or nilpotent (recall that $M_1 \in ind(A)$):
● If λ is invertible then:

$$\theta := \begin{bmatrix} \lambda & \mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \bigoplus \overline{M} \to M_1 \bigoplus \overline{M}$$

is invertible. Using (i) we deduce an isomorphism of exact sequences:

Since $h_1 : X \to M_1$ is homogeneous, (ε') fits property $(\mathcal{P})$. So $(\mathcal{P})$ is satisfied in this case.
● If $\lambda \in End_A(M_1)$ is nilpotent, let $p \geqslant 0$ be such that $\lambda^p = 0$. Using (i) we get the following equalities:

$$
\begin{aligned}
h_1 &= \lambda^2 h_1 + (\lambda^2 + \lambda)\overline{h} + (\lambda + Id_{M_1})\mu\overline{u} \\
\vdots \quad &\vdots \quad \vdots \\
h_1 &= \lambda^t h_1 + (\lambda^t + \lambda^{t-1} + \ldots + \lambda)\overline{h} + (\lambda^{t-1} + \ldots + \lambda + Id_{M_1})\mu\overline{u} \\
\vdots \quad &\vdots \quad \vdots \\
h_1 &= \lambda^p h_1 + (\lambda^p + \lambda^{p-1} + \ldots + \lambda)\overline{h} + (\lambda^{p-1} + \ldots + \lambda + Id_{M_1})\mu\overline{u}
\end{aligned}
$$

Since $\lambda^p = 0$ and $u_1 = h_1 + \overline{h}$ we infer that:

$$u_1 = \lambda'\overline{h} + \lambda'\mu\overline{u}$$

where $\lambda' := Id_{M_1} + \lambda + \ldots + \lambda^{p-1} \in End_A(M_1)$ is invertible. So we have an isomorphism:

$$\theta := \begin{bmatrix} \lambda' & \lambda'\mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \bigoplus \overline{M} \to M_1 \bigoplus \overline{M}$$

and consequently we have an isomorphism of exact sequences:

where $\overline{h} = h_2 + \ldots + h_p$ is the sum of $p-1$ non zero homogeneous morphisms of pairwise different degrees. So $(\mathcal{P})$ is satisfied in this case. This finishes the proof of the lemma. ∎

Now we can prove that $Y \in mod_1(A)$. Thanks to the preceding lemma, and with the same notations, we know that (ε) is isomorphic to an exact sequence in $mod(A)$:

$$0 \to X \xrightarrow{\begin{bmatrix} F_\lambda u_1' \\ \vdots \\ F_\lambda u_t' \end{bmatrix}} M_1 \bigoplus \ldots M_t \to Y \to 0 \qquad (\varepsilon')$$

where $u_i' \in Hom_\mathcal{C}(\widehat{X}, {}^{g_i}\widehat{M_i})$ for some $g_i \in G$, for every i. Therefore (recall that F_λ is exact):

$$Y \simeq Coker \begin{bmatrix} F_\lambda(u_1') \\ \vdots \\ F_\lambda(u_t') \end{bmatrix} \simeq F_\lambda \left(Coker \begin{bmatrix} u_1' \\ \vdots \\ u_t' \end{bmatrix} \right)$$

This proves that $Y \in mod_1(A)$. Therefore $T' = Y \bigoplus \overline{T} \in mod_1(A)$.

The proof of the implication $T' \in mod_1(A) \Rightarrow T \in mod_1(A)$ is similar, excepted that instead of using Lemma 3.2 we use a dual version:

Lemma 3.3. *Let $\varepsilon \colon 0 \to X \to M \xrightarrow{v} Y \to 0$ be an exact sequence in $mod(A)$ verifying the following hypotheses:*

. $X, Y \in ind(A)$ and $Y = F_\lambda \widehat{Y}$ (with $\widehat{Y} \in ind(\mathcal{C})$),

. $M = M_1 \bigoplus \ldots \bigoplus M_t$ where $M_i = F_\lambda \widehat{M_i} \in ind(A)$ (with $\widehat{M_i} \in ind(\mathcal{C})$), for every i,

. $Ext_A^1(M, X) = 0$.

Then (ε) is isomorphic to an exact sequence in $mod(A)$:

$$0 \to X \to M_1 \bigoplus \ldots \bigoplus M_T \xrightarrow{\begin{bmatrix} F_\lambda v_1' \\ \vdots \\ F_\lambda v_t' \end{bmatrix}} Y \to 0$$

where $v_i' \in Hom_\mathcal{C}({}^{g_i}\widehat{M_i}, \widehat{Y})$ for some $g_i \in G$, for every i.

This finishes the proof of Proposition 3.1. ∎

Remark 3.4. *Proposition 3.1 is similar to part of [12, Thm 3.6] where P. Gabriel proves the following: if $F \colon \mathcal{C} \to A$ is a Galois covering with group G, with $\mathcal{C}$ locally bounded and such that G acts freely on $ind(\mathcal{C})$, then for any connected component C of the Auslander-Reiten quiver of A, all indecomposable modules of C lie in $ind_1(A)$ as soon as any one of them does.*

Remark 3.5. *The proof of Proposition 3.1 shows that for an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_A$ such that $T, T' \in mod_1(A)$ there exists an exact sequence in $mod(\mathcal{C})$:*

$$0 \to X \xrightarrow{\iota} M \xrightarrow{\pi} Y \to 0$$

with the following properties:

. $T = F_\lambda X \bigoplus \overline{T}$ and $F_\lambda X \in ind(A)$,

. $T' = F_\lambda Y \bigoplus \overline{T}$ and $F_\lambda Y \in ind(A)$,

. $F_\lambda M \in add(\overline{T})$.

Recall that $\overrightarrow{\mathcal{K}}_A$ has a Brauer-Thrall type property (see [17, Cor. 2.2]): $\overrightarrow{\mathcal{K}}_A$ is finite and connected if it has a finite connected component. In particular, $\overrightarrow{\mathcal{K}}_A$ is finite and connected if A is of finite representation type. Using Proposition 3.1, we get the following corollary.

Corollary 3.6. *If $\overrightarrow{\mathcal{K}}_A$ is finite (e.g. A is of finite representation type), then any $T \in \overrightarrow{\mathcal{K}}_A$ is of the first kind w.r.t. F.*

Now we turn to the second goal of this section: for $T \in mod_1(A)$ a basic tilting A-module, give sufficient conditions for $F.T$ to be a basic $\mathcal{C}$-module.

Proposition 3.7. *Let $T, T' \in \overrightarrow{\mathcal{K}}_A \cap mod_1(A)$ lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$, then:*

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$$F.T \text{ is a basic } \mathcal{C}\text{-module} \Leftrightarrow F.T' \text{ is a basic } \mathcal{C}\text{-module}.$$

In particular, if $T' = A$ or $T' = DA$, then $T \in mod_1(A)$ and $F.T$ is a basic $\mathcal{C}$-module.

Proof: The k-category $\mathcal{C}$ is locally bounded so $F.A \simeq \mathcal{C}$ and $F.(DA) \simeq D\mathcal{C}$ are basic $\mathcal{C}$-modules. Therefore, we only need to prove the equivalence of the proposition. Without loss of generality, we may assume that there is an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_A$. Let us assume that $F.T$ is basic and let us prove that so is $F.T'$. We will use Remark 3.5 from which the adopt the notations, in particular, the exact sequence $0 \to X \xrightarrow{\iota} M \xrightarrow{\pi} Y \to 0$ in $mod(\mathcal{C})$ will be denoted by (ε). Because $F.T$ is basic and because of the properties verified by (ε), we only need to prove that $G_Y = 1$. Let $\varphi \colon Y \to {}^gY$ be an isomorphism in $mod(\mathcal{C})$ (with $g \in G$), and let us prove that $g = 1$. To do this we will exhibit an isomorphism $\theta \colon X \to {}^gX$. Notice that:

$$(\forall h \in G) \quad \begin{cases} {}^hX, \ {}^hM \in add(F.T) \\ {}^hY, \ {}^hM \in add(F.T') \end{cases} \tag{1}$$

Moreover, thanks to $T \in \overrightarrow{\mathcal{K}}_A$ and to $F_\lambda F.T = \bigoplus_{h \in G} T$ (which is true because T is of the first kind w.r.t. F, see Section 1), we have:

$$(\forall i \geqslant 1) \quad Ext^i_{\mathcal{C}}(F.T, F.T) \simeq Ext^i_A(F_\lambda F.(T), T) \simeq \prod_{h \in G} Ext^i_A(T, T) = 0$$

In particular:

$$Ext^1_{\mathcal{C}}({}^gM, X) = Ext^1_{\mathcal{C}}(M, {}^gX) = 0 \tag{i}$$

With $Hom_{\mathcal{C}}(M, {}^g\varepsilon)$, this last equality gives the exact sequence:

$$Hom_{\mathcal{C}}(M, {}^gM) \xrightarrow{({}^g\pi)_*} Hom_{\mathcal{C}}(M, {}^gY) \to Ext^1_{\mathcal{C}}(M, {}^gX) = 0$$

From this exact sequence, we deduce the existence of $\psi \in Hom_{\mathcal{C}}(M, {}^gM)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\pi} & Y \\
\downarrow{\psi} & & \downarrow{\varphi} \\
{}^gM & \xrightarrow{{}^g\pi} & {}^gY
\end{array}
$$

This implies the existence of $\theta \in Hom_{\mathcal{C}}(X, {}^gX)$ making commutative the following diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\
& & \downarrow{\theta} & & \downarrow{\psi} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & {}^gX & \xrightarrow{{}^g\iota} & {}^gM & \xrightarrow{{}^g\pi} & {}^gY & \longrightarrow & 0
\end{array} \tag{ii}
$$

We claim that $\theta \colon X \to {}^gX$ is an isomorphism. The arguments that have been used to get (ii) may be adapted (just replace the use of $Hom_{\mathcal{C}}(M, {}^g\varepsilon)$ and of $\varphi \colon Y \to {}^gY$ by $Hom_{\mathcal{C}}({}^gM, \varepsilon)$ and $\varphi^{-1} \colon {}^gY \to Y$) to get the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}^gX & \xrightarrow{{}^g\iota} & {}^gM & \xrightarrow{{}^g\pi} & {}^gY & \longrightarrow & 0 \\
& & \downarrow{\theta'} & & \downarrow{\psi'} & & \downarrow{\varphi^{-1}} & & \\
0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0
\end{array} \tag{iii}
$$

In order to show that $\theta \colon X \to {}^gX$ is an isomorphism, let us show that $\theta'\theta \in End_{\mathcal{C}}(X)$ is an isomorphism. Notice (ii) and (iii) give the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\
& & \downarrow{\theta'\theta - id_X} & & \downarrow{\psi'\psi - Id_M} & & \downarrow{0} & & \\
0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0
\end{array} \tag{iv}
$$

In particular we have $\pi(\psi'\psi - Id_M) = 0$, so there exists $\lambda \in Hom_{\mathcal{C}}(M, X)$ such that:

$$\psi'\psi - Id_M = \iota\lambda$$

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Therefore:
$$\iota(\theta'\theta - Id_X) = \iota\lambda\iota$$
Since ι is one-to-one, we get $\theta'\theta - id_X = \lambda\iota$, i.e.:
$$\theta'\theta = Id_X + \lambda\iota$$

If $\lambda\iota \in End_{\mathcal{C}}(X)$ was an isomorphism, then $\iota\colon X \to M$ would be a section. This would imply that $F_\lambda X$ is a direct summand of $F_\lambda M$. This last property is impossible because: $T = F_\lambda X \bigoplus \overline{T}$, $F_\lambda M \in add(\overline{T})$ and T is basic. This contradiction proves that $\lambda\iota \in End_{\mathcal{C}}(X)$ is nilpotent. Therefore $\theta'\theta = Id_X + \lambda\iota \in End_{\mathcal{C}}(X)$ is invertible. As a consequence, $\theta\colon X \to {}^g X$ is a section. Since $X, {}^g X \in ind(\mathcal{C})$, we deduce that $\theta\colon X \to {}^g X$ is an isomorphism. But we assumed that $F.T$ is basic, so $g = 1$. This finishes the proof of the implication:
$$F.T \text{ is basic} \Rightarrow F.T' \text{ is basic.}$$
After exchanging the roles of T and T' in the above arguments, we also prove that:
$$F.T \text{ is basic} \Leftarrow F.T' \text{ is basic.}$$
under the assumption that $T \to T'$ is an arrow in $\overrightarrow{\mathcal{K}}_A$. This achieves the proof of the proposition. ∎

Proposition 3.7 has the following corollary which will be useful to prove Theorem 1.

Corollary 3.8. *Let $F\colon \mathcal{C} \to A$ be a connected Galois covering with group G. Let $T = T_1 \oplus \ldots \oplus T_n$ ($T_i \in ind(A)$) and $T' = T'_1 \oplus \ldots \oplus T'_n$ ($T'_i \in ind(A)$) be basic tilting A-modules lying in a same connected component of $\overrightarrow{\mathcal{K}}_A$. Assume that $T, T' \in mod_1(A)$ and fix isomorphisms $\lambda_i\colon F_\lambda \widehat{T}_i \xrightarrow{\sim} T_i$ and $\lambda_i\colon F_\lambda \widehat{T'}_i \xrightarrow{\sim} T'_i$ with $\widehat{T}_i, \widehat{T'}_i \in ind(\mathcal{C})$ for every i. Then:*
$$F_{\widehat{T}_i, \lambda_i} \text{ is connected} \Leftrightarrow F_{\widehat{T'}_i, \lambda'_i} \text{ is connected.}$$
In particular, if $T' = A$ or $T' = DA$, then $F_{\widehat{T}_i, \lambda_i}$ is connected.

Proof: Recall that "the Galois covering $\mathcal{E} \to \mathcal{B}$ is connected" means $\mathcal{E}$ is connected and locally bounded. Thanks to Remark 2.1 and to Proposition 2.10, we know that $F_{\widehat{T}_i, \lambda_i}$ (resp. $F_{\widehat{T'}_i, \lambda'_i}$) is connected if and only if $F.T$ (resp. $F.T'$) is a basic $\mathcal{C}$-module. The corollary is therefore a consequence of Proposition 3.7. ∎

We finish with the last objective of the section: give "simple" conditions on $T \in \overrightarrow{\mathcal{K}}_A$ under which condition $(H_{A,T})$ is satisfied (see Lemma 2.5).

Proposition 3.9. *Let T, T' lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$. Then:*
$$(H_{A,T}) \text{ is satisfied} \Leftrightarrow (H_{A,T'}) \text{ is satisfied}$$
In particular, if $T' = A$ or $T' = DA$, then $(H_{A,T})$ is satisfied.

Proof: Let $\psi\colon A \xrightarrow{\sim} A$ be an automorphism restricting to the identity map on objects. Let $x \in Ob(A)$ and let $_?A_x\colon y \in Ob(A) \mapsto {}_yA_x$ be the indecomposable projective A-module associated to x. Then, we have an isomorphism of A-modules:
$$\begin{array}{rcl} _?A_x & \longrightarrow & \psi._?A_x \\ u \in {}_yA_x & \longmapsto & F(u) \in (\psi._?A_x)(y) = {}_yA_x \end{array}$$

So $(H_{A,A})$ is satisfied, and similarly $(H_{A,DA})$ is satisfied. Therefore, in order to prove the proposition, it suffices to prove that $(H_{A,T})$ is satisfied if and only if $(H_{A,T'})$ is satisfied, for any arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_A$. Assume that $T \to T'$ is an arrow in $\overrightarrow{\mathcal{K}}_A$ and that $(H_{A,T})$ is satisfied. So we have the data:

- $T = X \oplus \overline{T}$ with $X \in ind(A)$,
- $T' = Y \oplus \overline{T}$ with $Y \in ind(A)$,
- a non split exact sequence $0 \to X \xrightarrow{u} M \to Y \to 0$ where $M \in add(\overline{T})$ and where $u\colon X \to M$ is the left $add(\overline{T})$-approximation of X.

Notice that in order to prove that $(H_{A,T'})$ is satisfied, we only need to prove that $\psi.Y \simeq Y$. Since ψ is an automorphism, $\psi.\colon mod(A) \to mod(A)$ is an equivalence of abelian categories. Therefore, the sequence $0 \to \psi.X \xrightarrow{\psi.u} \psi.M \to \psi.Y \to 0$ is non split exact and verifies: $\psi.M \in add(\psi.\overline{T})$ and $\psi.u\colon \psi.X \to \psi.M$ is the left $add(\psi.\overline{T})$-approximation of $\psi.Y$. Moreover $\psi.X \simeq X$, $\psi.M \simeq M$ and $\psi.\overline{T} \simeq \overline{T}$ because $(H_{A,T})$ is satisfied. So, $\psi.Y$ is isomorphic to the cokernel of the left $add(\overline{T})$-approximation of X. This implies that $Y \simeq \psi.Y$. So $(H_{A,T'})$ is satisfied. The converse is dealt with using dual arguments. ∎

4 Comparison of $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_{End_A(T)}$ for a tilting A-module T

Let T be a basic tilting A-module. Let $B = End_A(T)$. In the preceding section, we have pointed out conditions of the form: "T lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A". Since our final objective (i.e. to compare the Galois coverings of A and of B) is symmetrical between A and B, we ought to find sufficient conditions for T to lie in both connected components of $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$ containing A and B respectively. Thus, this section is devoted to compare $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$. For simplicity, if $X \in mod(A)$ (resp. $u \in Hom_A(X,Y)$) we shall write X_T (resp. u_T) for the B-module (resp. the morphism of B-modules) $Hom_A(X,T)$ (resp. $Hom_A(u,T) \colon Hom_A(Y,T) \to Hom_A(X,T)$). Also, whenever f is a morphism of modules, we shall write f_* (resp. f^*) for the mapping $g \mapsto fg$ (resp. $g \mapsto gf$). We begin with a useful lemma.

Lemma 4.1. *Let* $X \in mod(A)$ *and let* $T' \in \overrightarrow{\mathcal{K}}_A$ *be a predecessor of* T *(i.e. there is an oriented path in* $\overrightarrow{\mathcal{K}}_A$ *starting at* T' *and ending at* T*). Then, the is an isomorphism, for any* $Y \in add(T')$:

$$\theta_{X,Y} \colon \quad Hom_A(X,Y) \quad \longrightarrow \quad Hom_B(Y_T, X_T)$$
$$u \quad \longmapsto \quad u_T$$

In particular: $Y \in ind(A) \Leftrightarrow Y_T \in ind(B)$, *for any* $Y \in add(T')$.

Proof: Remark that $\theta_{X,T'}$ is an isomorphism if and only if $\theta_{X,Y}$ is an isomorphism for any $Y \in add(T')$. By assumption on T', there exists a path in $\overrightarrow{\mathcal{K}}_A$ starting at T' and ending at T. Let us prove by induction on the length l of this path that $\theta_{X,T'}$ is an isomorphism.

If $l = 0$ then $T = T'$. So $\theta_{X,T'} = \theta_{X,T}$ is equal to:

$$Hom_A(X,T) = X_T \quad \longrightarrow \quad Hom_B(T_T, X_T) = Hom_B(B, X_T)$$
$$u \quad \longmapsto \quad (f \mapsto fu)$$

So $\theta_{X,T'}$ is an isomorphism (with inverse $\varphi \mapsto \varphi(1_B)$). This proves the lemma when $l = 0$.

Now, assume that $l > 0$ and assume that $\theta_{X,T''}$ is an isomorphism whenever T'' is the source of a path in $\overrightarrow{\mathcal{K}}_A$ ending at T and with length equal to $l-1$. We have a path $T' \to T'' \to \dots \to T$ of length l in $\overrightarrow{\mathcal{K}}_A$. Therefore:

$$\theta_{X,Y} \text{ is an isomorphism for any } Y \in add(T'') \tag{i}$$

Moreover, thanks to the arrow $T' \to T''$ in $\overrightarrow{\mathcal{K}}_A$, we have:

(ii) $T' = \overline{T} \bigoplus Y'$ with $Y' \in ind(A)$,

(iii) $T'' = \overline{T} \bigoplus Y''$ with $Y'' \in ind(A)$,

(iv) a non split exact sequence $0 \to Y' \to M \to Y'' \to 0$ with $M \in add(\overline{T})$.

Thanks to *(i)*, *(ii)* and *(iii)* we only need to prove that $\theta_{X,Y'}$ is an isomorphism. Remark that by assumption on T' and T'' we have $T \in T^\perp \subseteq T''^\perp$. This implies in particular that $Ext_A^1(Y'',T) = 0$. Therefore, *(iv)* yields an exact sequence in $mod(A)$:

$$0 \to Y''_T \to M_T \to Y'_T \to 0$$

This gives rise to the exact sequence:

$$0 \to Hom_B(Y'_T, X_T) \to Hom_B(M_T, X_T) \to Hom_B(Y''_T, X_T)$$

On the other hand, *(iv)* yields the following exact sequence:

$$0 \to Hom_A(X,Y') \to Hom_A(X,M) \to Hom_A(X,Y'')$$

Therefore, we have a commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Hom_A(X,Y') & \longrightarrow & Hom_A(X,M) & \longrightarrow & Hom_A(X,Y'') \\
& & \downarrow{\scriptstyle\theta_{X,Y'}} & & \downarrow{\scriptstyle\theta_{X,M}} & & \downarrow{\scriptstyle\theta_{X,Y''}} \\
0 & \longrightarrow & Hom_A(Y'_T, X_T) & \longrightarrow & Hom_A(M_T, X_T) & \longrightarrow & Hom_A(Y''_T, X_T)
\end{array}
$$

where the rows are exact and where $\theta_{X,M}$ and $\theta_{X,Y''}$ are isomorphisms. This shows that $\theta_{X,Y'}$ is an isomorphism. So $\theta_{X,T'}$ is an isomorphism and the induction is finished. This proves the first assertion of the lemma. The second assertion is due to the functoriality of $\theta_{X,Y}$. ∎

Remark 4.2. *Assume that A is hereditary. Then Lemma 4.1 still holds if one replaces the hypothesis "T' is a predecessor of T" by "$T' \geqslant T$" (i.e. $T^{\perp} \subseteq T'^{\perp}$). The proof is then a classical application of left $add(T)$-approximations.*

The following proposition is the base of the link between $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$: it explains how to associate suitable tilting B-modules with tilting A-modules.

Proposition 4.3. *Let $X \to Y$ be an arrow in $\overrightarrow{\mathcal{K}}_A$ where X and Y are predecessors of T. Then:*

$$X_T \in \overrightarrow{\mathcal{K}}_B \Leftrightarrow Y_T \in \overrightarrow{\mathcal{K}}_A$$

If the two conditions of the above equivalence are satisfied, then there is an arrow $Y_T \to X_T$ in $\overrightarrow{\mathcal{K}}_B$.

Proof: Let us assume that $Y_T \in \overrightarrow{\mathcal{K}}_B$ and let us show that $X_T \in \overrightarrow{\mathcal{K}}_B$ and that there is an arrow $Y_T \to X_T$ in $\overrightarrow{\mathcal{K}}_B$ (the proof of the remaining implication is then obtained by exchanging the roles of X and Y). The arrow $X \to Y$ in $\overrightarrow{\mathcal{K}}_A$ gives the following data:

. $X = M \bigoplus \overline{X}$ with $M \in ind(A)$,

. $Y = N \bigoplus \overline{X}$ with $N \in ind(A)$,

. $\varepsilon \colon 0 \to M \xrightarrow{i} X' \xrightarrow{p} N \to 0$ is a non split exact sequence in $mod(A)$ with $X' \in add(\overline{X})$.

The tilting A-module Y is a predecessor of T. Hence $T \in T^{\perp} \subseteq Y^{\perp}$ and therefore $Ext_A^1(N,T) = 0$. We infer that $Hom_A(\varepsilon, T)$ gives an exact sequence in $mod(B)$:

$$0 \to N_T \xrightarrow{p_T} X'_T \xrightarrow{i_T} M_T \to 0 \qquad\qquad (\varepsilon_T)$$

Notice that we also have:

. $X_T = M_T \bigoplus \overline{X}_T$,

. $Y_T = N_T \bigoplus \overline{X}_T$,

. $X'_T \in add(\overline{X}_T)$.

Therefore, in order to prove that $X_T \in \overrightarrow{\mathcal{K}}_B$ and that there is an arrow $Y_T \to X_T$ in $\overrightarrow{\mathcal{K}}_B$, we only need to prove the following facts:

1) ε_T does not split,

2) $M_T \in ind(B)$ and $N_T \in ind(B)$,

3) $pd_B(X_T) < \infty$,

4) X_T is selforthogonal,

5) X_T is the direct sum of n indecomposable A-modules and X_T is basic.

1) Let us prove that ε_T does not split. If ε_T splits, then i_T is a retraction:

$$(\exists \lambda \in Hom_B(M_T, X'_T)) \ \ Id_{M_T} = i_T \circ \lambda$$

Since M is a direct summand of $X \in \overrightarrow{\mathcal{K}}_A$ and since X is a predecessor of T, Lemma 4.1 implies that $\lambda = \pi_T$ with $\pi \in Hom_A(X', M)$. Thus we have $(\pi \circ i)_T = (Id_M)_T$. Using again Lemma 4.1 we deduce that $\pi \circ i = Id_M$ which is impossible because ε does not split. So ε_T does not split.

2) Lemma 4.1 implies that $M_T, N_T \in ind(B)$.

3) Since we assumed that $Y_T \in \overrightarrow{\mathcal{K}}_B$, we have $pd_B(\overline{X}_T) < \infty$, $pd_B(X'_T) < \infty$ and $pd_B(N_T) < \infty$. Hence ε_T gives $pd_B(M_T) < \infty$. So $pd_B(X_T) < \infty$.

4) Let us prove that X_T is selforthogonal. In this purpose, we will use the following lemma.

Lemma 4.4. *Let $L \in add(X)$. Then, the following morphism induced by $p_T \colon N_T \to X'_T$ is surjective:*

$$
\begin{array}{cccc}
(p_T)^* & Hom_B(X'_T, L_T) & \longrightarrow & Hom_B(N_T, L_T) \\
& f & \longmapsto & f \circ p_T
\end{array}
$$

Proof: Since $L \in add(X)$ and since $X \in \overrightarrow{\mathcal{K}}_A$, we have $Ext_A^1(L, M) = 0$. Hence, $Hom_A(L, \varepsilon)$ gives rise to a surjective morphism induced by p:

$$
\begin{array}{cccc}
p_* \colon & Hom_A(L, X') & \twoheadrightarrow & Hom_A(L, N) \\
& f & \longmapsto & p \circ f
\end{array}
$$

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Let us apply Lemma 4.1 to $X' \in add(Y)$ and to $N \in add(Y)$. We get the following commutative diagram where vertical arrows are isomorphisms:

$$
\begin{array}{ccc}
Hom_A(L, X') & \xrightarrow{\ p_* \ } & Hom_A(L, N) \\
{\scriptstyle \theta_{L,X'}}\downarrow & & \downarrow{\scriptstyle \theta_{L,N}} \\
Hom_B(X'_T, L_T) & \xrightarrow{\ (p_T)^* \ } & Hom_B(N_T, L_T)
\end{array}
$$

Since p_* is surjective, we infer that so is $(p_T)^*$. ∎

Now we can prove that $X_T = \overline{X}_T \bigoplus M_T$ is selforthogonal. Since $\overline{X}_T \in add(Y_T)$ and $Y_T \in \overrightarrow{\mathcal{K}}_B$, we get:

$$(\forall i \geqslant 1) \quad Ext^i_B(\overline{X}_T, \overline{X}_T) = 0 \tag{i}$$

For each $i \geqslant 1$, $Hom_B(\overline{X}_T, \varepsilon_T)$ gives the following exact sequence:

$$Ext^i_B(\overline{X}_T, X'_T) \to Ext^i_B(\overline{X}_T, M_T) \to Ext^{i+1}_B(\overline{X}_T, N_T)$$

Since $\overline{X}_T, X'_T, N_T \in add(Y_T)$ and $Y_T \in \overrightarrow{\mathcal{K}}_B$, we get:

$$(\forall i \geqslant 1) \quad Ext^i_B(\overline{X}_T, M_T) = 0 \tag{ii}$$

On the other hand, $Hom_B(\varepsilon_T, \overline{X}_T)$ gives the following exact sequences:

$$.Hom_B(X'_T, \overline{X}_T) \xrightarrow{(p_T)^*} Hom_B(N_T, \overline{X}_T) \to Ext^1_B(M_T, \overline{X}_T) \to Ext^1_B(X'_T, \overline{X}_T)$$

$$.Ext^i_B(N_T, \overline{X}_T) \to Ext^{i+1}_B(M_T, \overline{X}_T) \to Ext^{i+1}_B(X'_T, \overline{X}_T) \ \text{ for } i \geqslant 1$$

These exact sequences together with Lemma 4.4 and the selforthogonality of Y_T imply that:

$$(\forall i \geqslant 1) \quad Ext^i_B(M_T, \overline{X}_T) = 0 \tag{iii}$$

In order to get the selforthogonality of $X_T = M_T \bigoplus \overline{X}_T$ it only remains to prove that M_T is selforthogonal (because of $(i), (ii)$ and (iii)). Notice that $Hom_B(N_T, \varepsilon_T)$ gives the following exact sequence for each $i \geqslant 1$:

$$Ext^i_B(N_T, X'_T) \to Ext^i_B(N_T, M_T) \to Ext^{i+1}_B(N_T, N_T)$$

Using $Y_T \in \overrightarrow{\mathcal{K}}_B$ and $X'_T, N_T \in add(Y_T)$ we deduce that:

$$(\forall i \geqslant 1) \quad Ext^i_B(N_T, M_T) = 0 \tag{iv}$$

Finally $Hom_B(\varepsilon_T, M_T)$ gives the following exact sequences:

$$.Hom_B(X'_T, M_T) \xrightarrow{(p_T)^*} Hom_B(N_T, M_T) \to Ext^1_B(M_T, M_T) \to Ext^1_B(X'_T, M_T)$$

$$.Ext^i_B(N_T, M_T) \to Ext^{i+1}_B(M_T, M_T) \to Ext^{i+1}_B(X'_T, M_T) \ \text{ for } i \geqslant 1$$

These exact sequences together with Lemma 4.4, (ii) and (iv) imply that (recall that $X'_T \in add(\overline{X}_T)$):

$$(\forall i \geqslant 1) \quad Ext^i_B(M_T, M_T) = 0 \tag{v}$$

From (i), (ii), (iii) and (v) we deduce that $X_T = M_T \bigoplus \overline{X}_T$ is selforthogonal.

5) To finish, let us prove that X_T is basic and that X_T is the direct sum of n indecomposable modules. Notice that $\overline{X}_T$ is basic because it is a direct summand of the basic tilting B-module Y_T. On the other hand, ε_T does not split, so $Ext^1_A(M_T, N_T) \neq 0$, hence $M_T \notin add(Y_T)$ and therefore $M_T \notin add(\overline{X}_T)$. Since $M_T \in ind(B)$, we deduce that X_T is basic. Finally, Y_T is by assumption the direct sum of n indecomposable modules, and X_T and Y_T differ by one indecomposable direct summand so X_T is also the direct sum of n indecomposable modules. ∎

Remark 4.5. *When A is hereditary, Proposition 4.3 has the following generalisation: Let $X \in \overrightarrow{\mathcal{K}}_A$ be such that $X \geqslant T$, then $X_T \in \overrightarrow{\mathcal{K}}_B$. The proof of this generalisation is obtained by replacing the use of the exact sequence ε by a coresolution of X in $add(T)$.*

Proposition 4.3 gives the following proposition which will be used in the comparison of the Galois coverings of A and B. We omit the proof which is immediate using Proposition 4.3.

Proposition 4.6. *Let $X \in \overrightarrow{\mathcal{K}}_A$ be such that there exists a path in $\overrightarrow{\mathcal{K}}_A$ starting at X and ending at T. Then $X_T \in \overrightarrow{\mathcal{K}}_B$ and there exists in $\overrightarrow{\mathcal{K}}_B$ a path starting at B and ending at X_T.*

Proposition 4.3 also allows us to prove the main result of this section. Recall that for a quiver Q, we write Q^{op} for the opposite quiver (obtained from Q by reversing the arrows).

Theorem 4.7. *Let $\overrightarrow{\mathcal{K}}_A(T)$ (resp. $\overrightarrow{\mathcal{K}}_B(T)$) be the convex hull of $\{A,T\}$ (resp. $\{B,T\}$) in $\overrightarrow{\mathcal{K}}_A$ (resp. $\overrightarrow{\mathcal{K}}_B$). Then we have an isomorphism of quivers:*

$$\alpha: \quad \begin{array}{ccc} \overrightarrow{\mathcal{K}}_A(T) & \longmapsto & \overrightarrow{\mathcal{K}}_B(T)^{op} \\ X & \longmapsto & X_T = Hom_A(X,T) \end{array}$$

Under this correspondence, $A \in \overrightarrow{\mathcal{K}}_A(T)$ (resp. $T \in \overrightarrow{\mathcal{K}}_A(T)$) is associated with $T \in \overrightarrow{\mathcal{K}}_B(T)$ (resp. $B \in \overrightarrow{\mathcal{K}}_B(T)$).

Proof: Thanks to Proposition 4.3, the mapping α is a well defined morphism of quivers. Thus, it only remains to exhibit an inverse morphism. Notice that Proposition 4.6 implies that $\overrightarrow{\mathcal{K}}_A(T) = \{A,T\}$ $\Leftrightarrow$ $\overrightarrow{\mathcal{K}}_B(T) = \{B,T\}$ $\Leftrightarrow$ there is no path in $\overrightarrow{\mathcal{K}}_A(T)$ starting at A and ending at T. Therefore, we may assume that there is a path starting at A and ending at T. This assumption implies that any $X \in \overrightarrow{\mathcal{K}}_A(T)$ is a predecessor of T. From [21, Thm 1.5] we know that T is a basic tilting $End_B(T)$-module and that we have an isomorphism of k-algebras:

$$\begin{array}{ccc} A & \longrightarrow & End_B(T) \\ a & \longmapsto & (t \mapsto at) \end{array}$$

Henceforth, we shall consider A-modules as $End_B(T)$-modules and vice-versa using the above isomorphism. In particular, we have an identification of quivers:

$$\begin{array}{ccc} \overrightarrow{\mathcal{K}}_A(T) & \xrightarrow{\sim} & \overrightarrow{\mathcal{K}}_{End_B(T)}(T) \\ X & \mapsto & X \end{array}$$

Therefore, we also have a well defined morphism of quivers:

$$\alpha': \quad \begin{array}{ccc} \overrightarrow{\mathcal{K}}_B(T)^{op} & \to & \overrightarrow{\mathcal{K}}_A(T) \\ X & \mapsto & X_T = Hom_B(X,T) \end{array}$$

Let us prove that $\alpha'\alpha$ is an isomorphism. Let $X \in \overrightarrow{\mathcal{K}}_A(T)$. Then X is a predecessor of T. Therefore, Lemma 4.1 implies that:

$$Hom_B(Hom_A(X,T),T) \simeq Hom_B(Hom_A(X,T), Hom_A(A,T)) \simeq Hom_A(A,X) \simeq X$$

This proves that $\alpha'\alpha$ is an isomorphism of quivers. With the same arguments one also shows that $\alpha\alpha'$ is an isomorphism. So does $\alpha: \overrightarrow{\mathcal{K}}_A(T) \to \overrightarrow{\mathcal{K}}_B(T)^{op}$. ∎

Notice that $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B^{op}$ are not isomorphic in general. Indeed these quivers may have different number of vertices as the following example shows.

Example 4.8. *Let Q be the quiver:*
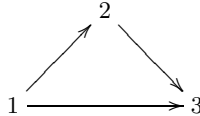
and let $A = kQ/I$ where I is the ideal generated by the oriented path of length 2 in Q. Notice that A is of finite representation type. Let $T = P_1 \oplus P_2 \oplus \tau_A^{-1} P_3$ be the APR-tilting A-module associated to the sink 3. Hence:

$$T = \begin{smallmatrix} 1 \\ 2\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1\ \ 2 \\ 2\ \ 3 \end{smallmatrix}$$

and the Hasse diagram $\overrightarrow{\mathcal{K}}_A$ of basic tilting A-modules is equal to:
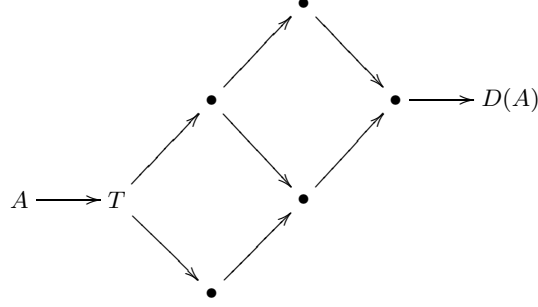
On the other hand, $B = End_A(T)$ is isomorphic to kQ'/I' where Q' is equal to:

$$a \longrightarrow b \rightleftharpoons c$$
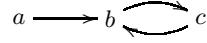
and I' is the ideal generated by the path $c \to b \to c$. As a B-module, T is equal to

$$T = \begin{matrix} a \\ b \\ c \\ b \end{matrix} \oplus \begin{matrix} c \\ b \end{matrix} \oplus \begin{matrix} a \ c \\ b \end{matrix}$$

and $\overrightarrow{\mathcal{K}}_B$ is equal to:
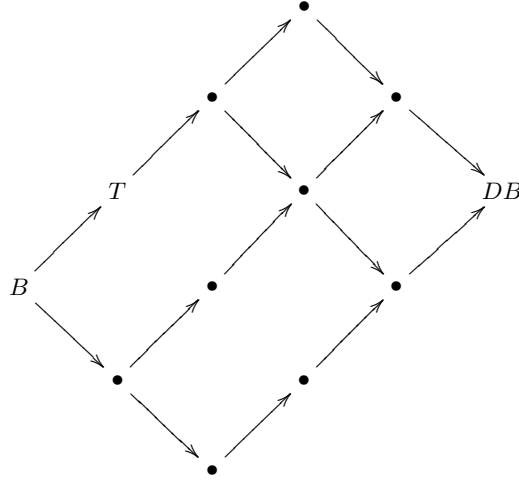
In particular, $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$ do not have the same number of vertices. Notice that, in this example, the isomorphism of Theorem 4.7 is equal to:

$$\begin{array}{rcl} \overrightarrow{\mathcal{K}}_A(T) = (A \to T) & \longrightarrow & \overrightarrow{\mathcal{K}}_B(T)^{op} = (T \to B) \\ A & \longmapsto & T = Hom_A(A,T) \\ T & \longmapsto & B = Hom_A(T,T) \end{array}$$

Remark 4.9. *Assume that A is hereditary, then Theorem 4.7 has the following generalisation, thanks to Remark 4.5: Let Q_A (resp. Q_B) be the full subquiver of $\overrightarrow{\mathcal{K}}_A$ (resp. $\overrightarrow{\mathcal{K}}_B$) made of the tilting modules $X \geqslant T$. Then $X \mapsto X_T$ induces an isomorphism of quivers $Q_A \xrightarrow{\sim} Q_B^{op}$.*

5 Comparison of the Galois coverings of A and $End_A(T)$ for T basic tilting A-module

This section is devoted to the proof of Theorem 1, of Corollary 1 and of Corollary 2. Let $T \in \overrightarrow{\mathcal{K}}_A$ and let $B = End_A(T)$. As in the introduction, we shall say that A and B have the same connected Galois coverings with group G if and only if there exists a bijection $Gal_A(G) \xrightarrow{\sim} Gal_B(G)$. Here $Gal_A(G)$ denotes

the set of equivalence classes of connected Galois coverings with group G of A. In order to compare the equivalence classes of connected Galois coverings of A and those of B, we introduce the following assertion which depends on A, on T and on a fixed group G:

$\mathcal{P}(A,T,G) =$"$(H_{A,T})$ is satisfied and for any connected Galois covering $F\colon \mathcal{C} \to A$ with group G, the A-module T is of the first kind w.r.t. F and $F.T$ is a basic $\mathcal{C}$-module"

Recall from Definition 2.7 that the condition $(H_{A,T})$ ensures the existence of an equivalence class $[F]_T$ of Galois coverings of B depending only on the equivalence class $[F]$ of F. Recall also from Remark 2.1 and from Proposition 2.10 that the condition "$F.T$ is a basic $\mathcal{C}$-module" implies that $[F]_T$ is an equivalence class of connected Galois coverings of B. Finally, recall that $\mathcal{P}(A,A,G)$ and $\mathcal{P}(A,DA,G)$ are true for any G (see Proposition 3.1, Proposition 3.7 and Proposition 3.9). The above definition of $\mathcal{P}(A,T,G)$ is relevant because of the following proposition.

Proposition 5.1. *Let G be a group. Assume that $\mathcal{P}(A,T,G)$ and $\mathcal{P}(B,T,G)$ are true. Then A and B have the same connected Galois coverings with group G.*

Proof: Since $\mathcal{P}(A,T,G)$ is true, we have a well defined mapping:

$$\varphi_A\colon \quad \begin{array}{ccc} Gal_A(G) & \longrightarrow & Gal_B(G) \\ [F] & \longmapsto & [F]_T \end{array} \qquad (i)$$

Similarly, $\mathcal{P}(B,T,G)$ is true so we have a well defined mapping:

$$\varphi_B\colon \quad \begin{array}{ccc} Gal_B(G) & \longrightarrow & Gal_G(End_B(T)) \\ [F] & \longmapsto & [F]_T \end{array} \qquad (ii)$$

Thanks to Proposition 2.11 we know that $\rho_A^{-1} \circ (\varphi_B\varphi_A([F])) = [F]$ for any $[F] \in Gal_A(G)$. Therefore, φ_A is one-to-one and φ_B is onto. Notice that thanks to the isomorphism $\rho_A\colon A \xrightarrow{\sim} End_B(T)$, the assertion $\mathcal{P}(End_B(T),T,G)$ is true, so that the above arguments imply that φ_B is one-to-one and that $\varphi_{End_B(T)}$ is onto. As a consequence, φ_B is bijective, so the mapping $[F] \mapsto \rho_A^{-1} \circ [F]_T$ induces a bijection $Gal_B(G) \xrightarrow{\sim} Gal_A(G)$. ∎

Thanks to Proposition 5.1 we are reduced to find sufficient conditions for $\mathcal{P}(A,T,G)$ and $\mathcal{P}(B,T,G)$ to be simultaneously true. The following proposition is a direct consequence of Proposition 3.1, of Corollary 3.8, of Proposition 3.9 and of the fact that $\mathcal{P}(A,A,G)$ and $\mathcal{P}(A,DA,G)$ are true.

Proposition 5.2. *Let G be a group. Let $T' \in \overrightarrow{\mathcal{K}}_A$ lying in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing T. Then:*

$$\mathcal{P}(A,T,G) \text{ is true} \Leftrightarrow \mathcal{P}(A,T',G) \text{ is true}$$

In particular, if $T' = A$ or if $T' = DA$ then $\mathcal{P}(A,T,G)$ is true.

Thanks to Proposition 5.2, we are reduced look for conditions for T to lie in both connected components of $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$ containing A and B respectively. Such a condition is given by the following proposition.

Proposition 5.3. *Let G be a group and assume that there exists a path in $\overrightarrow{\mathcal{K}}_A$ starting at A and ending at T. Then T lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ (resp. $\overrightarrow{\mathcal{K}}_B$) containing A (resp. B). Consequently, $\mathcal{P}(A,T,G)$ and $\mathcal{P}(B,T,G)$ are true.*

Proof: Theorem 4.7 implies that there exists a path in $\overrightarrow{\mathcal{K}}_B$ starting at $Hom_A(T,T) = B$ and ending at $Hom_A(A,T) = T$. Using Proposition 5.2 we get the desired conclusion. ∎

Now we can prove Theorem 1:

Proof of Theorem 1: 1) Since T and T' lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$, there exists a sequence $T^{(1)} = T, T^{(2)}, \dots, T^{(r)} = T'$ of basic tilting A-modules such that for any $i \in \{1, \dots, r-1\}$, there exists a path in $\overrightarrow{\mathcal{K}}_A$ with $T^{(i)}$ and $T^{(i+1)}$ as end-points. For short, let us write B_i for $End_A(T^{(i)})$. Let $i \in \{1, \dots, r-1\}$ and let us assume, for example, that there exists a path in $\overrightarrow{\mathcal{K}}_A$ starting at $T^{(i)}$ and ending at $T^{(i+1)}$. Using Lemma 4.1 and Proposition 4.6 we infer that:

(i) $End_A(T^{(i)})$ and $End_{B_{i+1}}(Hom_A(T^{(i)}, T^{(i+1)}))$ are isomorphic as k-algebras (and therefore as k-categories),

(ii) there exists a path in $\overrightarrow{\mathcal{K}}_{B_{i+1}}$ starting at B and ending at $Hom_A(T^{(i)}, T^{(i+1)})$.

This implies (thanks to Proposition 5.3 and to Proposition 5.1) that $End_A(T^{(i)})$ and $End_A(T^{(i+1)})$ have the same connected Galois coverings with group G. Since this fact is true for any i, we deduce that $End_A(T)$ and $End_A(T')$ have the same connected Galois coverings with group G.

2) is a consequence of 1), of the fact that $End_A(A) \simeq End_A(DA) \simeq A^{op}$ and of the fact that A and A^{op} have the same Galois coverings ($F: \mathcal{C} \to A$ is a Galois covering if and only if $F^{op}: \mathcal{C}^{op} \to A$ is a Galois covering and $\mathcal{C}^{op}$ is connected and locally bounded if and only if $\mathcal{C}^{op}$ is). ∎

Using Theorem 1 we can prove Corollary 1 and Corollary 2.

Proof of Corollary 1: Since A is of finite representation type, Theorem 1 implies that A and B have the same connected Galois coverings. For the same reason, A (resp. B) admits a connected Galois covering with group G if and only if G is a factor group of the fundamental group $\pi_1(A)$ (resp. $\pi_1(B)$) of A (resp. of B). Consequently, $\pi_1(A)$ and $\pi_1(B)$ are isomorphic. ∎

Proof of Corollary 2: 1) and 2) are consequences of Theorem 1 and of the fact that A is simply connected if and only if it has no proper connected Galois covering (see [19, Cor. 4]).

3) is a consequence of 2). ∎

Corollary 1 naturally leads to the following question: let G be a group such that A and B have the same Galois coverings with group G, is it true that A admits an admissible presentation with fundamental group isomorphic to G if and only if the same holds for B? The answer is no in general as the following example shows :

Example 5.4. *Let Q be the following quiver:*
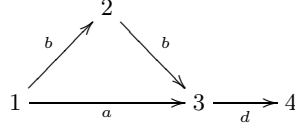
and let $A = kQ/I$ where $I =< da >$. Let $T = P_1 \oplus P_2 \oplus P_3 \oplus \tau_A^{-1}(P_4) = P_1 \oplus P_2 \oplus P_3 \oplus S_3$ be the APR-tilting module associated with the sink 3 (here S_i is the simple A-module associated to the vertex i and P_i is the indecomposable projective A-module with top S_i). Then $B = End_A(T)$ is the path algebra of the following quiver:
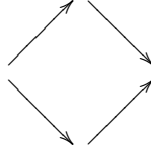
Since T is an APR-tilting A-module, there is an arrow $A \to T$ in $\overrightarrow{\mathcal{K}}_A$. Then, Theorem 1 implies that for any group G, the k-algebras A and B have the same connected Galois covering with group G. On the other hand, any admissible presentation of B has fundamental group isomorphic to $\mathbb{Z}$ whereas A admits an admissible presentation with fundamental group 0 and another one with fundamental group isomorphic to $\mathbb{Z}$ (see for example [4, 1.4]).

In the preceding example, the reader may remark that the fundamental group of any admissible presentation of A is a factor group of $\mathbb{Z}$ and that the same holds for B. Let us say that A admits an optimum fundamental group (G) if and only if there exists an admissible presentation of A with fundamental group G and if the fundamental group of any other admissible presentation is a factor group of G. For example, A admits an optimum fundamental group in the following cases: A is of finite representation type (see [12]), A is constricted (see [8, Thm 3.5]), A is monomial, A is triangular and has no double bypass (see [19, Thm. 1]). Then we have the following corollary whose proof is a direct consequence of Theorem 1:

Corollary 5.5. *Assume that T lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A. Then A admits G as optimum fundamental group if and only if B admits G as optimum fundamental group.*

6 On the simple connectedness of a tilted algebra

The aim of this section is to prove Proposition 1. We shall use the construction of the preceding section. Recall from Remark 2.1, Lemma 2.2 and Proposition 2.10 that given a tilting A-module T and a connected

Galois covering $F \colon \mathcal{C} \to A$, we need two properties in order to define a connected Galois covering of $End_A(T)$:

1. T is of the first w.r.t. F,

2. $F.T$ is a basic $\mathcal{C}$-module.

For the first property, we use the following lemma which is due to W. Crawley-Boevey (see [13]). I acknowledge B. Keller and C. Geiss who gave me the details about this lemma.

Lemma 6.1. *(W. Crawley-Boevey, [13]) Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is the characteristic of k. ($G = \mathbb{Z}$ if $car(k) = 0$). Let $\mathcal{E}, \mathcal{B}$ be locally bounded k-categories and let $F \colon \mathcal{E} \to \mathcal{B}$ be a Galois covering with group G. Let M be a rigid indecomposable $\mathcal{B}$-module. Then M is of the first kind w.r.t. F.*

Remark 6.2. *Lemma 6.1 also holds for any group in the class of groups containing $\mathbb{Z}/p\mathbb{Z}$ and closed under direct product, under quotient and under isormophism.*

Lemma 6.3. *Let G be a finite group, let $F \colon A' \to A$ be a connected Galois covering with group G and let T be a basic tilting A-module of the first kind w.r.t. F. Then $F.T$ is a basic A'-module.*

Proof: First, let us prove that $F.T$ is a basic A'-module. From Lemma 1.2 applied to T, we have $pd_{A'}(F.T) < \infty$. Besides, $F_\lambda F.T \simeq \oplus_{g \in G} T$ because T is of the first kind w.r.t. F, so that:

$$(\forall i \geqslant 1) \quad Ext^i_{A'}(F.T, F.T) \simeq Ext^i_A(F_\lambda F.T, T) \simeq Ext^i_A(\bigoplus_{g \in G} T, T) \simeq \bigoplus_{g \in G} Ext^i_A(T, T) = 0$$

Thus, $F.T$ is an exceptionnal A'-module. Moreover, if we apply $F.$ to any finite coresolution of A in $add(T)$, we deduce a finite coresolution of $A' \simeq F.A$ in $add(F.T)$. Hence $F.T$ is a tilting A'-module. Now we can prove that $F.T$ is basic. Since T is basic, it is the direct sum of n indecomposable A-modules. Moreover, T is of the first kind w.r.t. F, so $F.T$ is the direct sum of $n.|G| = rk(K_0(A'))$ indecomposable A'-modules. Since $F.T$ is a tilting A' module, this implies that $F.T$ is basic. ∎

Now we can prove Proposition 1.

Proof of Proposition 1: Recall (see for example [15]) that $HH^1(kQ) = 0$ if and only if Q is a tree. Now assume that Q is not a tree. Then, A admits a connected Galois covering $F \colon \mathcal{C} \to A$ with group $\mathbb{Z}$. Let G be any finite cyclic group if $car(k) = 0$ and let $G = \mathbb{Z}/p\mathbb{Z}$ if $car(k) = p$ is non zero. Let N be the kernel of the natural surjection $\mathbb{Z} \twoheadrightarrow G$. Then there exists a commutative diagram:

where $p \colon \mathcal{C} \to \mathcal{C}/N$ is the natural projection and $\overline{F} \colon \mathcal{C}/N \to A$ is a connected Galois covering with group G. Then:

- if $car(k) = 0$, then Lemma 6.1 implies that T is of the first kind w.r.t. F so that it is also of the first kind w.r.t. $\overline{F}$.

- if $car(k) = p$ is non zero then Lemma 6.1 implies that T is of the first kind w.r.t. $\overline{F}$.

Thanks to Lemma 6.3 we deduce that $\overline{F} \colon \mathcal{C}/N \to A$ is connected Galois covering of A with group G and verifying: T is of the first kind w.r.t. F and $F.T$ is a basic $\mathcal{C}/N$-module. Using Remark 2.1 and Proposition 2.10, we deduce that $End_A(T)$ admits a connected Galois covering with group $G \neq 1$. Hence $End_A(T)$ is not simply connected. ∎

Final remark

The Hasse diagram $\overrightarrow{\mathcal{K}}_A$ of basic tilting A-modules describes the combinatoric relations between tilting modules. When A is hereditary (i.e. $A = kQ$ with Q a finite quiver with no oriented cycle) these combinatorics are also described by the cluster category $\mathcal{C}_Q$ of the quiver Q (see [11]). In particular, the

indecomposable tilting objects in $\mathcal{C}_Q$ are displayed as the vertices of an unoriented graph. Since this graph is always connected (see [11, 3.5]) it is natural to ask if it is possible to remove all conditions concerning connected components in Theorem 1 and Corollary 2 (in the hereditary case). These developpements will be detailed in a forecoming text.

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